

THE QUOTIENT OF A COMPLETE SYMMETRIC VARIETY

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Dedicated to Ernest Vinberg on the occasion of his 70th birthday

1. INTRODUCTION

Let G be a semisimple simply connected algebraic group over an algebraically closed field of characteristic different from 2. Given an involution σ of G with fixed subgroup G^σ , we fix a subgroup $G^\sigma \subset H \subset N_G(G^\sigma)$.

Our goal in this paper is the study of the action of H on certain completions of G/H with the methods of geometric invariant theory.

The study of such problems starts with the famous paper [12] of Kostant and Rallis in which the H action on the quotient $\mathrm{Lie} G / \mathrm{Lie} H$ is studied. This can be considered as an infinitesimal version of our study. Results similar to those in [12] have been later obtained by Richardson in [14] in the case of the quotient G/H .

In particular Richardson has proved, among other things, that if we take the image S_H in G/H of an anisotropic maximal torus S in G and consider the action of the restricted Weyl group \tilde{W} on S_H (see below for the definitions), the GIT quotient $H \backslash\backslash G/H$ is isomorphic to $\tilde{W} \backslash S_H$. Furthermore he shows that the closed H orbits in G/H are precisely the orbits of elements in S_H .

In this paper we generalize these two results to the case of a completion Y of G/H . In particular we reprove the results of Richardson mentioned above.

To state our result take a smooth toroidal projective G equivariant completion Y of G/H . In Y consider the closure Y_S of S_H . The \tilde{W} action on S_H extends to an action on Y_S . Fix an ample line bundle \mathcal{L} on Y . Our first result is that the GIT quotient $H \backslash\backslash_{\mathcal{L}} Y$ relative to \mathcal{L} is isomorphic to $\tilde{W} \backslash Y_S$. In particular this quotient does not depend on the choice of \mathcal{L} . (Theorem 4.1).

We then pass to the study of the set $Y^{ss} \subset Y$ of semistable points in Y with respect to \mathcal{L} . Also in this case we show that Y^{ss} does not depend on the choice of \mathcal{L} (Remark 5.6) and we describe rather precisely the intersection of Y^{ss} with any G orbit. In particular we show that given two H orbits $O_1 \subset \overline{O_2}$ in Y^{ss} then they both lie in the same G orbit (Proposition 6.1) and that a H orbit O in Y^{ss} is closed if and only if it meets Y_S (Theorem

6.4) . These last facts allow us to give a version of our results in the case of any G stable open subset in Y .

The proofs of our results are rather straightforward in characteristic zero and are based on the careful analysis of sections of line bundles on Y given in [1] and [2]. However to carry out our proofs in positive characteristic we have to deal with a number of rather technical results which often do not appear in the literature and which, in view of this, we have decided to explain here.

2. PRELIMINARIES

In this section we introduce notations, recall some simple properties and describe the spherical weights relative to a given involution.

Let us choose an algebraically closed field \mathbb{k} whose characteristic is not equal to 2. Usually all algebraic group schemes in this paper are going to be affine and defined over \mathbb{k} but, occasionally we are going to consider group schemes defined over the ring $\mathbf{A} := \mathbb{Z}[1/2]$ and flat over $\mathrm{Spec} \mathbf{A}$. Gothic letters are going to denote Lie algebras.

Let G be a semisimple and simply connected algebraic group. Let \bar{G} be the adjoint quotient of G and Z the kernel of the projection of the isogeny $G \longrightarrow \bar{G}$. This is a possibly not reduced subgroup of G whose associated reduced subvariety is given by the center of G .

Let σ be an involution of G and let $H^\sigma = G^\sigma$ be the subgroup of elements fixed by σ . We consider also the inverse image \bar{H} under the isogeny $G \longrightarrow \bar{G}$ of the subgroup of \bar{G} of elements fixed by the involution of \bar{G} induced by σ . We recall that H^σ is connected and reductive and that \bar{H} is a possibly not reduced subgroup of G whose associated reduced subgroup is the normalizer $N_G(H^\sigma)$ of H^σ . It is known that the connected component of the identity of \bar{H} with reduced structure is equal to H^σ (see [4]).

Let now $H^\sigma \subset H \subset \bar{H}$ be a possibly not reduced subgroup of G . The quotient $G/H = \mathrm{Spec} \mathbb{k}[G]^H$ is called a *symmetric variety*.

We fix an *anisotropic maximal torus* S of G , that is a torus of G such that $\sigma(s) = s^{-1}$ for all $s \in S$, and having maximal dimension among the tori with this property. The dimension ℓ of S is called the *rank* of the symmetric variety. We choose also a σ stable maximal torus T of G containing S and a Borel subgroup containing T with the property that the intersection $B \cap \sigma(B)$ has minimal possible dimension. Occasionally we will also need to consider *isotropic tori*, that is tori contained in H .

2.1. Ring of definition. It will be important for us that the classification of involutions is independent of the characteristic (see [17]). So we can use Kac classification or Satake classification to construct the involutions. If we use Kac classification we see that we can assume that G, σ and hence H^σ are all defined over \mathbf{A} and that there is a maximally isotropic maximal torus (this means a maximal torus of G containing a maximal torus of H^σ) defined over \mathbf{A} and a σ stable Borel subgroup containing this torus also

defined over \mathbf{A} . On the other hand if we use Satake classification, we see that we can assume that G, σ, H° , the maximal torus T , the torus S and the Borel subgroup B are all defined over \mathbf{A} . However, occasionally, we will need to work with a \mathbf{A} form of G where both maximally isotropic and maximally anisotropic maximal tori are defined and split over \mathbf{A} .

We start with a flat \mathbf{A} form \mathbf{G} of G , and with a σ defined over \mathbf{A} constructed using Kac diagrams. So H° is defined over \mathbf{A} and there exists a \mathbf{A} split maximal torus N of H° defined over \mathbf{A} and a \mathbf{A} split maximal torus M of G defined over \mathbf{A} containing N . The characters of M, N are defined over \mathbf{A} and the root decomposition of the Lie algebra of G is also defined over \mathbf{A} . In particular all Borel subgroups containing M are defined over \mathbf{A} . Let B_M be a σ stable Borel subgroups of G containing M . Let Ψ and $\Psi^+ \subset \Psi$ be the corresponding sets of roots and positive roots of the Lie algebra of G with respect to B_M . Finally notice that also \bar{H} , hence H , can be assumed to be defined over \mathbf{A} .

We want to show that in G there is a maximally anisotropic maximal torus defined and split over \mathbf{A} . This slightly strength a result in [6].

Lemma 2.1. *There is a torus S in G defined and split over \mathbf{A} such that $\sigma(s) = s^{-1}$ for all $s \in S$ and S has maximal dimension among the tori with this property. Moreover there is a maximal torus T of G containing S defined and split over \mathbf{A} . Finally the root decomposition of \mathfrak{g} with respect to the action of the torus T is defined over \mathbf{A} and there is a Borel B subgroup containing T and defined and flat over \mathbf{A} such that the dimension of $\sigma(B) \cap B$ is the minimal possible. The two Borel subgroups B and B_M are conjugated by an element of $\mathbf{G}(\mathbf{A})$.*

Proof. For each root $\beta \in \Psi$ denote by $u_\beta(t)$ the corresponding one parameter subgroup. This subgroup can be defined over \mathbf{A} . We construct (see [11] §VI.7) the torus T as follows. Let $\mathcal{B} \subset \Psi^+$ be a set of roots maximal among the subsets with the following properties

- i) $\beta \in \mathcal{B}$ implies $\sigma(\beta) = \beta$ and $\sigma(u_\beta(t)) = u_\beta(-t)$;
- ii) $\beta, \beta' \in \mathcal{B}$ implies $\beta + \beta', \beta - \beta' \notin \Psi$.

For each $\beta \in \mathcal{B}$ set

$$g_\beta = u_\beta(1)u_{-\beta}(-1/2), \text{ and } g_{\mathcal{B}} = \prod_{\beta \in \mathcal{B}} g_\beta.$$

Notice that since by ii) the roots in \mathcal{B} are orthogonal to each other, the elements g_β as β runs in \mathcal{B} commute and $g_{\mathcal{B}}$ is well defined and lies in $\mathbf{G}(\mathbf{A})$.

We then set $T = g_{\mathcal{B}} M g_{\mathcal{B}}^{-1}$ and $S = g_{\mathcal{B}} M_{\mathcal{B}} g_{\mathcal{B}}^{-1}$ where $M_{\mathcal{B}}$ is the subtorus of T corresponding to the coroots in \mathcal{B} . By [11] §VI.7 T and S have all the required properties.

Our claims about the root decomposition now follows from the analogous properties for the torus M , and under this hypothesis it is clear that each Borel containing T is defined and flat over \mathbf{A} . Also notice that $g_{\mathcal{B}} B_M g_{\mathcal{B}}^{-1}$

is a Borel containing T so it must be conjugated to B by an element in $N_G(T)$. Now by Lemma 2.7 in [6] every element of the Weyl group has a representative in $\mathbf{G}(\mathbf{A})$ proving the last claim. \square

We finish this section with a simple Lemma regarding invariants. Since we are going to deal with non necessarily reduced algebraic groups let us recall that if L is a not necessarily reduced algebraic group and V is a representation of L , a L invariant vector $v \in V$ is a vector whose image under the coaction $V \longrightarrow V \otimes \mathbb{k}[L]$ is $v \otimes 1$.

Lemma 2.2. *Let \mathbf{L} be an algebraic group scheme defined and flat over \mathbf{A} (we do not assume that \mathbf{L} is either connected or reduced in general) and let V be a finite dimensional representation of \mathbf{L} defined and flat over \mathbf{A} . Assume that $V(\mathbb{C})^{\mathbf{L}(\mathbb{C})} \neq 0$. Then there is an \mathbf{L} invariant vector defined over \mathbf{A} whose reduction modulo p is different from 0 for all odd primes p . In particular $V(\mathbb{k})^{\mathbf{L}(\mathbb{k})} \neq 0$.*

Proof. Let $V_{\mathbf{A}}$ be an \mathbf{A} lattice compatible with the action. The action of \mathbf{L} on V is given by the coaction map $a^{\sharp} : V_{\mathbf{A}} \longrightarrow \mathbf{A}[\mathbf{L}] \otimes_{\mathbf{A}} V_{\mathbf{A}}$. If B is a \mathbf{A} algebra, set $a_B^{\sharp} = \text{id}_B \otimes_{\mathbf{A}} a^{\sharp}$. Thus an element v in $V(B) := B \otimes_{\mathbf{A}} V_{\mathbf{A}}$ is fixed by \mathbf{L} if $a_B^{\sharp}(v) = 1 \otimes v$. Let now $F : V_{\mathbf{A}} \longrightarrow \mathbf{A}[\mathbf{L}] \otimes_{\mathbf{A}} V_{\mathbf{A}}$ be given by $F(v) = a^{\sharp}(v) - 1 \otimes v$ and $F_B = \text{id}_B \otimes F$. $V(B)^{\mathbf{L}} = \ker F_B$. In particular notice that when B is a field of characteristic zero we have, since $V_{\mathbf{A}}$ is a free \mathbf{A} module, that $\ker F_B = B \otimes_{\mathbf{A}} \ker F$. In particular $V(\mathbb{C})^{\mathbf{L}(\mathbb{C})} = \mathbb{C} \otimes_{\mathbf{A}} \ker F$ is defined over \mathbf{A} . Moreover since also $\mathbf{A}[\mathbf{L}] \otimes_{\mathbf{A}} V_{\mathbf{A}}$ has no torsion, we have that if $n \in \mathbb{Z} \setminus \{0\}$, $v \in V_{\mathbf{A}}$ and $nv \in \ker F$ then $v \in \ker F$. So $\ker F$ is a direct summand of $V_{\mathbf{A}}$.

It follows that $\mathbb{k} \otimes_{\mathbf{A}} \ker F$ injects into a non zero subspace of $V(\mathbb{k})^{\mathbf{L}(\mathbb{k})}$ proving our claim. \square

2.2. Spherical weights and the restricted root system. We want to describe now the Weyl modules of G which have a non zero H invariant vector.

If A is a torus we denote with Λ_A its character lattice $\text{Hom}(A, \mathbb{k}^*)$. Given a surjective homomorphism $A \longrightarrow B$ between tori, we are going to consider Λ_B as a sublattice of Λ_A .

Let $\Lambda = \Lambda_T$ and let $r : \Lambda \longrightarrow \Lambda_S$ be the surjective homomorphism induced by the inclusion $S \subset T$. Let also Φ be the root system of \mathfrak{g} with respect to T , Φ^+ (resp. Δ) be the choice of positive roots (resp. the simple roots) corresponding to the Borel B and Λ^+ be the dominant weights with respect to B .

Every character λ of T extends uniquely to a one dimensional character of B and we define \mathcal{L}_{λ} as the line bundle $G \times_B \mathbb{k}_{-\lambda}$ on G/B . Every line bundle on G/B is isomorphic to a line bundle of this form. For $\lambda \in \Lambda^+$ the Weyl module V_{λ} is defined as the dual of the space of sections $\Gamma(G/B, \mathcal{L}_{\lambda})$. With the choices of the previous section, all these objects are defined over

A. Furthermore it is well known [10] that $\Gamma(G/B, \mathcal{L}_\lambda)$ and hence V_λ is flat over \mathbf{A} . Occasionally we will have to consider also line bundles on a partial flag variety G/P where P is a parabolic subgroup containing B . The natural projection $G/B \rightarrow G/P$ induces an inclusion of $\text{Pic}(G/P)$ in $\text{Pic}(G/B) = \Lambda$ and allows us to identify $\text{Pic}(G/P)$ with the sublattice Λ_P of Λ consisting of those characters λ of B which extend to P . For $\lambda \in \text{Pic}(G/P)$ we are going, by abuse of notation, to denote by \mathcal{L}_λ the line bundle $G \times_P \mathbb{k}_{-\lambda}$ on G/P .

We define the monoid of dominant H spherical weights as

$$\Omega_H^+ = \{\lambda \in \Lambda^+ : \Gamma(G/B, \mathcal{L}_\lambda)^H \neq 0\}$$

and the lattice of spherical weights Ω_H as the lattice generated by Ω_H^+ . We set also $\Omega = \Omega_{H^\circ}$ and $\Omega^+ = \Omega_{H^\circ}^+$.

Recall that, since H has an open orbit in G/B , if $\lambda \in \Lambda^+$ then the space of H invariant sections $\Gamma(G/B, \mathcal{L}_\lambda)^H$ is at most one dimensional. A non zero vector in $\Gamma(G/B, \mathcal{L}_\lambda)^H$ will be called a *spherical vector*.

Let us now give a description of Ω . In characteristic zero Ω has been described by Helgason [9] using analytic methods. An algebraic proof of these results has been given by Vust [19]. The Theorem of Vust is stated in characteristic zero but its proof can be used verbatim in any characteristic different from 2 once we replace V_λ with V_λ^* . Moreover Vust's proof can also be easily adapted to describe the lattice Ω_H . Let $S_H = S/S \cap H$ then we have the following Theorem.

Theorem 2.3 (Vust [19] Théorème 3). *Let $\lambda \in \Lambda^+$ then $\lambda \in \Omega_H^+$ if and only if $\sigma(\lambda) = -\lambda$ and $r(\lambda) \in \Lambda_{S_H}$.*

We will need also to study quasi invariants under the action of \bar{H} , so we define a dominant weight λ to be *quasi spherical* if the representation $\Gamma(G/B, \mathcal{L}_\lambda)$ has a line fixed by \bar{H} . We denote the monoid of quasi spherical dominant weights by Π^+ and we set Π equal to the sublattice spanned by Π^+ and call it the lattice of quasi spherical weights.

Quasi spherical weights have been described in terms of spherical weights and exceptional roots by De Concini and Springer in [6].

Let Φ_0 (resp. Δ_0) be the set of roots (resp. simple roots) fixed by σ and let Φ_1 (resp. Δ_1) be the complement of Φ_0 in Φ (resp. of Δ_0 in Δ). With our choices of the Borel subgroup B we have $\sigma(\alpha) \in \Phi^-$ for all $\alpha \in \Phi_1^+ = \Phi_1 \cap \Phi^+$ (see [4]). Moreover the involution σ induces an involution $\bar{\sigma}$ of Δ_1 where $\bar{\sigma}(\alpha)$ is the unique simple root such that $\sigma(\alpha) + \bar{\sigma}(\alpha)$ lies in the span of Δ_0 . A simple root $\alpha \in \Delta_1$ is said to be *exceptional* if $\bar{\sigma}(\alpha) \neq \alpha$ and $\kappa(\sigma(\alpha), \alpha) \neq 0$, κ being a non degenerate bilinear form on Λ invariant under the action of the Weyl group. We denote by $\{\omega_\alpha\}_{\alpha \in \Delta}$, the fundamental weights with respect to the simple basis Δ . We have,

Theorem 2.4 (De Concini and Springer [6] Lemma 4.6 and Theorem 4.8).

- i) For each $\lambda \in \Pi^+$ the line fixed by \bar{H} is unique.
- ii) Π^+ is generated as a monoid by Ω^+ and the fundamental weights ω_α corresponding to the exceptional roots.

The set of spherical weights is related to the *restricted root system* as follows. Let us quickly recall how restricted roots are defined. If $\alpha \in \Phi$ is not fixed by σ we define the *restricted root* $\tilde{\alpha}$ as $\alpha - \sigma(\alpha)$ and the restricted root system $\tilde{\Phi} \subset \Lambda$ as the set of all restricted roots. This is a (not necessarily reduced) root system (see [14]) of rank ℓ and the subset $\tilde{\Phi}^+$ (resp. $\tilde{\Delta}$) of restricted roots $\tilde{\alpha}$ with α positive (resp. α simple) is a choice of positive roots (resp. a simple basis) for $\tilde{\Phi}$.

We collect in the following lemma some known and easy consequences of the previous theorems.

Lemma 2.5.

- i) $\Pi \cap \Lambda^+ = \Pi^+$ and $\Omega_H \cap \Lambda^+ = \Omega_H^+$;
- ii) In the adjoint case we have $\Omega_{\tilde{H}} = \mathbb{Z}[\tilde{\Phi}]$;
- iii) In the simply connected case we have
$$\Omega = \{\lambda \in \Lambda : \sigma(\lambda) = -\lambda \text{ and } \frac{2\kappa(\lambda, \tilde{\alpha})}{\kappa(\tilde{\alpha}, \tilde{\alpha})} \in \mathbb{Z} \text{ for all } \tilde{\alpha} \in \tilde{\Phi}\};$$
- iv) If $\lambda \in \Lambda$ and $n\lambda \in \Omega$ for some positive natural number n then $\sigma(\lambda) = -\lambda$;
- v) The restriction of r to Ω_H is injective and $r(\Omega_H) = \Lambda_{S_H}$.

In particular, by iii), Ω^+ is the set of dominant weights of the root system $\tilde{\Phi}$, so it is a free monoid of rank ℓ and a basis of it is given by fundamental weights $\tilde{\omega}_{\tilde{\alpha}}$ with respect to $\tilde{\Delta}$. Notice that if α is exceptional also $\beta = \bar{\sigma}(\alpha)$ is exceptional. If this is the case, we shall call $\tilde{\alpha} \in \tilde{\Delta}$ an exceptional restricted simple root and we recall that $\tilde{\omega}_{\tilde{\alpha}} = \omega_{\alpha} + \omega_{\beta}$.

Finally we apply Lemma 2.2 to our situation.

Corollary 2.6. *If $\lambda \in \Omega_H^+$ then V_{λ} has a nonzero vector fixed by H and if $\lambda \in \Pi^+$ then V_{λ} has a line fixed by \tilde{H} . More precisely there is a vector of V_{λ} defined over \mathbf{A} whose reduction modulo any odd prime is different from 0 and fixed by H (respectively spans a line fixed by \tilde{H}).*

Proof. Let G, σ, V_{λ} be defined over \mathbf{A} as explained above. Let M, N, B_M be as in section 2.1. In particular any character of the group H^o is a character of N hence it is defined over \mathbf{A} .

Let now $\lambda \in \Omega_H^+$. Since $V_{\lambda}(\mathbb{C})$ contains a non zero vector fixed by H the claim follows from Lemma 2.2.

In general notice that since H^o is a spherical subgroup (it has an open orbit in G/B) it acts on two different lines in $V_{\lambda}(\mathbb{C})$ stabilized by H^o with different characters. In particular any line in $V_{\lambda}(\mathbb{C})$ which is stabilized by $H^o(\mathbb{C})$ must be defined over \mathbf{A} : indeed let R be such a line and consider the character χ of H^o given by the action of H^o on R . Recall that with our choices all characters of H^o are defined over \mathbf{A} . By applying Lemma 2.2 to $V_{\lambda} \otimes \chi^{-1}$ we see that the line R must be defined over \mathbf{A} . In particular the line stabilized by $\tilde{H}(\mathbb{C})$ in $V_{\lambda}(\mathbb{C})$ is stabilized by H^o so it is defined over \mathbf{A} and it is \tilde{H} stable. \square

2.3. Line bundles on G/H . In this section we want to study some properties of the line bundles on G/H . We begin with a remark on \bar{H}/H .

Lemma 2.7. *The coordinate ring of \bar{H}/H is isomorphic to the group algebra $\mathbb{k}[\Omega_H/\Omega_{\bar{H}}]$.*

Proof. Let $H \cap S = H \times_G S$ be the scheme theoretic intersection of H and S . By Proposition 7 in [19] we have $H = H^o \cdot (H \cap S)$. So we have

$$\bar{H}/H \simeq \bar{H} \cap S/H \cap S \simeq \ker\{S_H \twoheadrightarrow S_{\bar{H}}\}.$$

where the kernel has to be considered scheme theoretically. Now by Lemma 2.5 v) we have $S_{\bar{H}} \simeq \text{Spec } \mathbb{k}[\Omega_{\bar{H}}]$ and $S_H \simeq \text{Spec } \mathbb{k}[\Omega_H]$.

It follows that, if we denote by e^χ the function on S_H corresponding to $\chi \in \Omega_H$, the coordinate ring of the kernel is then given by $\mathbb{k}[S_H]/\langle e^\chi - 1 : \chi \in \Omega_{\bar{H}} \rangle \simeq \mathbb{k}[\Omega_H/\Omega_{\bar{H}}]$, proving the claim. \square

We denote by x_H the point of G/H corresponding to the coset eH and by $q_H : G/H \rightarrow G/\bar{H}$ the projection induced by inclusion $H \subset \bar{H}$.

The line bundles on G/H are parametrized by the set of one dimensional characters Λ_H of H by associating to a line bundle \mathcal{L} the character by which H acts on the fiber of \mathcal{L} over x_H .

If $\lambda \in \Pi^+$ by Theorem 2.4 the line fixed by \bar{H} in V_λ^* is unique and we can consider the character $-\chi_H(\lambda)$ given by the action of H on this line. The map $\chi_H : \Pi^+ \rightarrow \Lambda_H$ extends to a group homomorphism $\chi_H : \Pi \rightarrow \Lambda_H$ and by Lemma 2.5 i) the kernel of this homomorphism is given by Ω_H . In particular for any $\xi \in \Pi/\Omega_{\bar{H}}$ we can consider a line bundle \mathcal{L}_ξ on G/\bar{H} whose associated isomorphism class is given by $\chi_{\bar{H}}(\xi)$.

Proposition 2.8. *The vector bundles $(q_H)_*(\mathcal{O}_{G/H})$ and $\bigoplus_{\xi \in \Omega_H/\Omega_{\bar{H}}} \mathcal{L}_\xi$ on G/\bar{H} are G equivariantly isomorphic.*

Proof. Set $\Xi_H = \Omega_H/\Omega_{\bar{H}}$. Notice first that by Lemma 2.7 the map q_H is a covering of degree equal to the cardinality of Ξ_H . So the two vector bundles $(q_H)_*(\mathcal{O}_{G/H})$ and $\bigoplus_{\xi \in \Xi_H} \mathcal{L}_\xi$ have the same rank.

If $\xi \in \Xi_H$ then $q_H^*(\mathcal{L}_\xi)$ is trivial for all ξ as a G linearized line bundle. So, by adjunction, we have a G equivariant monomorphism of sheaves $\mathcal{L}_\xi \rightarrow (q_H)_*(\mathcal{O}_{G/H})$. We deduce for any subset $R \subset \Xi_H$ a G equivariant map $\gamma_R : \bigoplus_{\xi \in R} \mathcal{L}_\xi \rightarrow (q_H)_*(\mathcal{O}_{G/H})$. Since γ_R is equivariant the induced map at the level of the total spaces of vector bundles has constant rank.

We claim that γ_R is of rank $|R|$. If $|R| = 1$ this is clear by the above considerations. We proceed by induction. Write $R = R' \cup \{\xi\}$. $\gamma_{R'}$ is of rank $|R| - 1$. Assume γ_R is not of maximal rank. We clearly get an inclusion $j : \mathcal{L}_\xi \rightarrow \bigoplus_{\xi' \in R'} \mathcal{L}_{\xi'}$. In particular there exists $\xi' \in R'$ such that the composition of j with the projection onto $\mathcal{L}_{\xi'}$ is a non zero G equivariant morphism and thus an isomorphism of line bundles. Since $\xi \neq \xi'$ this is a contradiction.

If we apply this to $R = \Xi_H$ and use the fact that $(q_H)_*(\mathcal{O}_{G/H})$ and $\bigoplus_{\xi \in \Xi_H} \mathcal{L}_\xi$ have the same rank we get that γ_{Ξ_H} is an isomorphism as desired. \square

3. COMPLETIONS OF SYMMETRIC VARIETIES

An *embedding* of a symmetric variety G/H is a normal connected G variety Y together with an open G equivariant inclusion $j_Y : G/H \subset Y$. We set y_0 equal to the image of x_H under this embedding and call it the basepoint of Y . We are also going to consider the finite covering $\pi_Y : G/H^\circ \rightarrow Y$ of j_Y given by $\pi_Y(gH^\circ) = g \cdot y_0$. We denote by Y_0 the image of j_Y and set $\partial Y = Y \setminus Y_0$ and Δ_Y equal to the set of irreducible components of ∂Y of codimension 1 in Y .

A line bundle \mathcal{L} on Y is said to be *spherical* if $\pi_Y^*(\mathcal{L})$ is isomorphic to the trivial line bundle on G/H° . We denote $\text{SPic}(Y)$ the subgroup of the Picard group $\text{Pic}(Y)$ of Y of spherical line bundles. We also say that a line bundle is *strictly spherical* if restricted to the open orbit G/H it is isomorphic to the trivial line bundle and we denote by $\text{SPic}_0(Y)$ the subgroup of $\text{Pic}(Y)$ of classes of strictly spherical line bundles.

Many of the properties of Y can be deduced from corresponding properties of the associated toric variety Y_S . This is defined as the closure of the orbit $S \cdot y_0$ in Y . Notice that since $S \cdot y_0$ is isomorphic to S_H , Y_S is a toric variety for the torus S_H . The normalizer $N_{H^\circ}(S)$ of S in H° acts on Y_S and the action of the centralizer $Z_{H^\circ}(S)$ of S is trivial. It follows that we have an action of the *restricted Weyl group* $\tilde{W} = N_{H^\circ}(S)/Z_{H^\circ}(S)$ on Y_S .

We are now going to describe an open subvariety Y_S^+ of Y_S with the property the \tilde{W} translates of Y_S^+ cover Y_S . Let Λ_S^\vee be the lattice of one parameter subgroups of S . If $\eta \in \Lambda_S^\vee$ and there exists the limit $\lim_{t \rightarrow \infty} \eta(t) \cdot y_0$ we denote this limit by y_η . We say that $\eta \in \Lambda_S^\vee$ is *positive* if $\tilde{\alpha}(\eta(t))$ is a non negative power of t for all $\tilde{\alpha} \in \tilde{\Delta}$ and let Y_S^+ the union of the S orbits of the elements $\{y_\eta : \eta \in \Lambda_S^\vee \text{ is positive}\}$. It is then immediate to verify that $Y_S = \tilde{W}Y_S^+$. Indeed if $y \in Y_S$ there is a $\eta \in \Lambda_S^\vee$ and a $s \in S$ such that $y = s(\lim_{t \rightarrow \infty} \eta(t) \cdot y_0)$. Since η is \tilde{W} conjugate to a positive one parameter subgroup we deduce y is \tilde{W} conjugate to an element in Y_S^+ .

3.1. The wonderful compactification of a symmetric variety. The so called wonderful compactification X of the symmetric variety G/\bar{H} has been introduced in characteristic zero in [4] and in arbitrary characteristic in [6]. We want to very briefly recall some of the basic properties of X and introduce some notations.

Recall that by Lemma 2.5 and Theorem 2.3 a basis of the character lattice $\Lambda_{S_{\bar{H}}}$ is given by the set $\tilde{\Delta} = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_\ell\}$ of simple restricted roots (with an arbitrarily chosen numbering). Thus we get an action, defined over \mathbf{A} , of $S_{\bar{H}}$ on the affine space \mathbb{A}^ℓ given by $s(a_1, \dots, a_\ell) = (\tilde{\alpha}_1(s)a_1, \dots, \tilde{\alpha}_\ell(s)a_\ell)$. The

following Theorem can be taken implicitly as the definition of the wonderful compactification.

Theorem 3.1 (Theorem 3.1 in [4], Proposition 3.10, Theorem 3.10 and Theorem 3.13 in [6]). *The wonderful compactification X of G/\bar{H} is the unique G/\bar{H} embedding such that*

- i) *X is a smooth projective G variety and the closure of every G orbit in X is smooth.*
- ii) *∂X is a divisor with normal crossing and smooth irreducible components.*
- iii) *given a G orbit $\mathcal{O} \subset X$, $\bar{\mathcal{O}}$ is the transversal intersection of the irreducible divisors in Δ_X containing it.*
- iv) *The intersection of any number of divisors in Δ_X is a G orbit closure. In particular the intersection of all divisors in Δ_X is the unique closed G orbit in X .*
- v) *There exists a scheme \mathbf{X} defined and flat over \mathbf{A} whose specialization to \mathbb{k} is isomorphic to X . Moreover the point $x_0 = j_X(x_{\bar{H}})$ is defined over \mathbf{A} .*
- vi) *Let \mathbf{G} be as in Section 2.1. There is an action of \mathbf{G} on \mathbf{X} that specializes over \mathbb{k} to the action of G on X .*
- vii) *We have an isomorphism $X_S^+ \simeq \mathbb{A}^\ell$ as S_{H° toric varieties defined over \mathbf{A} .*

Since any projective G variety is isomorphic to a variety G/P with P a parabolic subgroup containing B , we have already remarked that its Picard group can be identified with a sublattice of Λ . Thus composing with the homomorphism induced by the inclusion of the unique closed orbit, we get a homomorphism $j : \text{Pic}(X) \rightarrow \Lambda$. One has,

Theorem 3.2 (Theorem 4.2 and Theorem 4.8 in [6]).

- i) *The homomorphism j is injective and its image is the sublattice Π of Λ .*
- ii) *The map $D \rightarrow j(\mathcal{O}(D))$ is a bijection between Δ_X and $\tilde{\Delta}$.*

Notice that combining these two results we easily see that we get a bijection between the subsets $\Gamma \subset \tilde{\Delta}$ and the set of G orbit closures defined by associating to Γ the intersection

$$X_\Gamma := \cap_{\{D | j(\mathcal{O}(D)) \in \Gamma\}} D.$$

In particular $X_{\tilde{\Delta}}$ is the unique closed orbit while $X = X_\emptyset$. Let also $X_{\tilde{\alpha}} = X_{\{\tilde{\alpha}\}}$ for $\tilde{\alpha} \in \tilde{\Delta}$.

For each $\lambda \in \Pi$ we choose a line bundle \mathcal{L}_λ on X such that $j(\mathcal{L}_\lambda) = \lambda$ in the following way. First we choose a basis \mathcal{B} of Π and for each $\beta \in \mathcal{B}$ we take a line bundle with the required property. Now, for $\lambda = \sum_{\beta \in \mathcal{B}} c_\beta \beta \in \Pi$, $c_\beta \in \mathbb{Z}$, we set $\mathcal{L}_\lambda := \bigotimes_{\beta \in \mathcal{B}} \mathcal{L}_\beta^{\otimes c_\beta}$. We denote the restriction of these line bundles to $X_{\tilde{\Delta}}$ by the same symbol.

If $L \subset \Lambda$ is a sublattice of Λ , then our definition allows us to consider the graded rings

$$R_L(X) := \bigoplus_{\lambda \in L} \Gamma(X, \mathcal{L}_\lambda) \quad \text{and} \quad R_L(X_{\tilde{\Delta}}) := \bigoplus_{\lambda \in L} \Gamma(X_{\tilde{\Delta}}, \mathcal{L}_\lambda).$$

The ring $R(X) = R_\Pi(X)$ it is called the *Cox ring* of X and it was studied in the case of the variety X in [3] where it was called the ring of all sections. The fact that G is simply connected implies that each line bundle on X has a canonical G linearization. It follows G acts on $R_L(X)$ and $R_L(X_{\tilde{\Delta}})$.

The space $\Gamma(X, \mathcal{L}_\lambda)$ of sections of \mathcal{L}_λ has been described as a G module in [4] and [6]. Let us recall here this description.

Recall that a good filtration of a G module W is a filtration $W = W_0 \supset W_1 \supset \dots \supset W_m = \{0\}$ by G submodules such that for each $i = 1, \dots, m$, W_{i-1}/W_i is isomorphic to $\Gamma(G/B, \mathcal{L}_{\lambda_i})$ for a suitable dominant weight λ_i .

The result in [6] implies that $\Gamma(X, \mathcal{L}_\lambda)$ has a good filtration. To be more precise first of all one shows that for any $\lambda \in \Pi$ the map

$$\Gamma(X, \mathcal{L}_\lambda) \rightarrow \Gamma(X_{\tilde{\Delta}}, \mathcal{L}_\lambda)$$

is surjective.

Now for any $\lambda, \mu \in \Pi$ set $\mu \leq_\sigma \lambda$ if $\lambda - \mu \in \mathbb{N}[\tilde{\Delta}]$.

Notice that, for $\tilde{\alpha} \in \tilde{\Delta}$, there is a G invariant section $s_{\tilde{\alpha}}$ of $\mathcal{L}_{\tilde{\alpha}}$, unique up to multiplication by a non zero scalar, whose divisor is $X_{\tilde{\alpha}}$.

If $\nu = \sum_{\tilde{\alpha}} n_{\tilde{\alpha}} \tilde{\alpha} \geq_\sigma 0$ consider $s^\nu := \prod_{\tilde{\alpha}} s_{\tilde{\alpha}}^{n_{\tilde{\alpha}}}$. If $\lambda \geq_\sigma \mu$, the multiplication by $s^{\lambda-\mu}$ defines a G equivariant injective map from $\Gamma(X, \mathcal{L}_\mu)$ to $\Gamma(X, \mathcal{L}_\lambda)$ whose image we denote by $s^{\lambda-\mu} \Gamma(X, \mathcal{L}_\mu)$.

For any $\nu \geq_\sigma 0$ we now set

$$F_{\lambda, \nu} = \sum_{\mu \leq_\sigma \lambda - \nu} s^{\lambda-\mu} \Gamma(X, \mathcal{L}_\mu).$$

The $F_{\lambda, \nu}$ form a decreasing filtration of $\Gamma(X, \mathcal{L}_\lambda)$ by G submodules. In [4, 6] the associated graded is computed and we have that the division by s^ν and restriction of sections to $X_{\tilde{\Delta}}$ gives an isomorphism $F_{\lambda, \nu} / (\sum_{\nu' >_\sigma \nu} F_{\lambda, \nu'}) \simeq V_{\lambda-\nu}^*$ so that

$$(1) \quad \text{Gr}_F \Gamma(X, \mathcal{L}_\lambda) = \bigoplus_{\mu \in \Pi^+ \text{ and } \mu \leq_\sigma \lambda} s^{\lambda-\mu} V_\mu^*.$$

Clearly the filtration $F_{*,*}$ respects multiplication. This implies that the associated graded

$$\text{Gr}_F R(X) := \bigoplus_{\lambda \in \Pi} \text{Gr}_F \Gamma(X, \mathcal{L}_\lambda)$$

of $R(X)$ has a ring structure. Furthermore (1) gives a ring isomorphism

$$(2) \quad \text{Gr}_F R(X) \simeq R_\Pi(X_{\tilde{\Delta}})[s_{\tilde{\alpha}_1}, \dots, s_{\tilde{\alpha}_\ell}].$$

In the previous section we have studied spherical weights. We want to prove now that λ is spherical precisely when \mathcal{L}_λ is spherical.

The homomorphism $\pi_X^* : \text{Pic}(X) \rightarrow \text{Pic}(G/H^o)$ can be identified with the homomorphism $\chi : \Pi \rightarrow \Lambda_{H^o}$ associating to $\lambda \in \Pi$, the character $\chi(\lambda)$ by which H^o acts on the fiber of \mathcal{L}_λ on the point x_0 .

We claim that $\chi(\lambda) = \chi_{H^o}(\lambda)$ is the dual of the character by which H^o acts on the line fixed by \bar{H} in V_λ^* introduced in the previous section. To see this we may assume $\lambda \in \Pi^+$.

We fix $\lambda \in \Pi^+$ and $\mathcal{L} = \mathcal{L}_\lambda$. In this case \mathcal{L} has no base points over $X_{\bar{\Delta}}$, so, since $X_{\bar{\Delta}}$ is the unique closed orbit in X , by Theorem 3.1 *iv*) it also has no base points over X . Thus by the reductivity of \bar{H} , there is a positive integer m and a line $L \subset \Gamma(X, \mathcal{L}^m)$ stable under the action of \bar{H} and such that if $\sigma \in L - \{0\}$, σ does not vanish on x_0 . It follows that H^o acts on L by the character $-m\chi(\lambda)$.

Take the filtration $\{F_{m\lambda, \nu}\}$ of $\Gamma(X, \mathcal{L}^m)$. There is a unique submodule $F_{m\lambda, \nu}$ such that $L \subset F_{m\lambda, \nu} - \sum_{\nu' >_\sigma \nu} F_{m\lambda, \nu'}$. So L has non zero image in $V_{m\lambda - \nu}^*$ and thus coincides with the unique \bar{H} stable line in $V_{m\lambda - \nu}^*$. We deduce that $m\chi(\lambda) = \chi_{H^o}(m\lambda - \nu)$. Since ν lies in $\mathbb{Z}[\tilde{\Phi}]$, we have $\chi_{H^o}(m\lambda - \nu) = \chi_{H^o}(m\lambda)$, hence $m\chi_{H^o}(\lambda) = m\chi(\lambda)$. Finally since H^o is connected its character group has no torsion and we get that $\chi_{H^o}(\lambda) = \chi(\lambda)$ as desired. We deduce the following lemma.

Lemma 3.3. *Let $\lambda \in \Pi$ then $\pi_X^*(\mathcal{L}_\lambda)$ is trivial if and only if $\lambda \in \Omega$. Moreover if $\pi_H : G/H \rightarrow X$ is defined by $\pi_H(gH) = g \cdot x_0$ then $\pi_H^*(\mathcal{L}_\lambda)$ is trivial if and only if $\lambda \in \Omega_H$.*

Proof. The first claim has just been proved. As for the second it follows since by Theorem 2.3 a character λ lies in $\Omega_H \cap \Pi^+$ if and only if the line in V_λ^* stable under \bar{H} is pointwise invariant under H . \square

3.2. Toroidal compactifications and ring of definition. An embedding Y of G/H is called *toroidal* if there exists a basepoint preserving G equivariant map $\phi : Y \rightarrow X$.

Presently we are going to explain their construction and show that they are defined and flat over \mathbf{A} .

Let $L_{\mathbb{R}} = \text{Hom}(\Lambda_S, \mathbb{R})$ and $L_{\mathbb{R}}^\vee = \Lambda_S \otimes_{\mathbb{Z}} \mathbb{R}$ be its dual. The S , or S_H , toric varieties are described by fans in $L_{\mathbb{R}}$. In particular take the co-chamber $C \subset L_{\mathbb{R}}$ of dominant elements with respect to $\bar{\Delta}$ and let \mathcal{T}_H be the S_H toric variety associated to C . \mathcal{T}_H has a natural \mathbf{A} form \mathcal{T}_H . In particular in the adjoint case $\mathcal{T}_{\bar{H}} \simeq \mathbb{A}_{\mathbf{A}}^\ell$.

Choose a \mathbf{A} form of G as in Section 2.1. Consider for any H the finite field extension $\mathbb{Q}(G/\bar{H}) \subset \mathbb{Q}(G/H)$. $\mathbb{Q}(G/\bar{H})$ is the field of rational functions on \mathbf{X} and we take \mathbf{X}_H equal to the normalization of \mathbf{X} in $\mathbb{Q}(G/H)$. Let $\phi_H : \mathbf{X}_H \rightarrow \mathbf{X}$ denote the normalization map and let $X_H = \mathbf{X}_H(\mathbb{k})$.

Lemma 3.4. *\mathbf{X}_H is a projective normal and Cohen-Macaulay embedding of G/H . ϕ_H is a finite flat morphism. In particular \mathbf{X}_H is proper and flat over \mathbf{A} .*

Proof. The projectivity and normality of \mathbf{X}_H are clear from the definitions. Let us show that \mathbf{X}_H is Cohen-Macaulay.

To see this, let us recall X is covered by the G translates of an open set \mathcal{U} of the form $X_S^+ \times U$ where U is the unipotent radical of the parabolic $P \subset B$ such that $X_{\tilde{\Delta}} \simeq G/P$. By Theorem 3.1 we have that X_S^+ and U are defined over \mathbf{A} and so is the isomorphism $X_S^+ = \mathcal{T}_{\bar{H}} \simeq \mathbb{A}^\ell$. In particular the open set \mathcal{U} is defined over \mathbf{A} and we denote by \mathcal{U} the associated subscheme of \mathbf{X} and \mathbf{U} the subgroup scheme of \mathbf{G} defining U .

It easily follows that \mathbf{X}_H is covered by the $\mathbf{G}(\mathbf{A})$ translates of the preimage \mathcal{U}_H of \mathcal{U} and that $\mathcal{U} \simeq \mathcal{T}_H \times \mathbf{U}$. Since \mathcal{T}_H is Cohen-Macaulay, also \mathcal{U} is Cohen-Macaulay and everything follows.

Since any finite morphism between a Cohen-Macaulay scheme and a smooth scheme is flat we deduce that ϕ_H is flat and all the other claims are clear. \square

We are now going to follow the method of [5] to build all toroidal compactifications. For each $\tilde{\alpha} \in \tilde{\Delta}$ we have already chosen a line bundle $\mathcal{L}_{\tilde{\alpha}}$ on X together with a G invariant section $s_{\tilde{\alpha}} \in \Gamma(X, \mathcal{L}_{\tilde{\alpha}})$. We can then consider the vector bundle $\mathcal{V} := \bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} \mathcal{L}_{\tilde{\alpha}}$ and the G invariant section $s := \bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} s_{\tilde{\alpha}}$ of \mathcal{V} . Set $\mathcal{V}^* = \{v = (v_{\tilde{\alpha}}) \in \mathcal{V} : v_{\tilde{\alpha}} \neq 0 \ \forall \ \tilde{\alpha} \in \tilde{\Delta}\}$. By our previous identifications \mathcal{V}^* is a principal $S_{\bar{H}}$ bundle. If Z is a $S_{\bar{H}}$ variety we can take the associate bundle $\mathcal{V}^* \times_{S_{\bar{H}}} Z$ on X with fiber Z . In particular $\mathcal{V} = \mathcal{V}^* \times_{S_{\bar{H}}} \mathbb{A}^\ell$, where $S_{\bar{H}}$ acts on \mathbb{A}^ℓ via the characters $\tilde{\alpha} \in \tilde{\Delta}$.

Now take Z to be a $S_{\bar{H}}$ embedding over \mathbb{A}^ℓ . The corresponding fan F_Z is a (partial) decomposition of the fundamental Weyl chamber C . The map $Z \rightarrow \mathbb{A}^\ell$ induces a map $\mathcal{V}^* \times_{S_{\bar{H}}} Z \rightarrow \mathcal{V}$ and we define X_Z as the fiber product

$$\begin{array}{ccc} X_Z & \xrightarrow{s_Z} & \mathcal{V}^* \times_{S_{\bar{H}}} Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{s} & \mathcal{V} \end{array}$$

The G action on \mathcal{V} preserves \mathcal{V}^* and commutes with the $S_{\bar{H}}$ action. So G also acts on $\mathcal{V}^* \times_{S_{\bar{H}}} Z$, the map $\mathcal{V}^* \times_{S_{\bar{H}}} Z \rightarrow \mathcal{V}$ is G equivariant and X_Z is a G variety.

In the case of a general G/H we set $X_{H,Z}$ equal to the normalization of X_Z in the field of rational function on G/H . We clearly have the cartesian diagram

$$\begin{array}{ccc} X_{H,Z} & \xrightarrow{\mu_Z} & X_Z \\ \downarrow & & \downarrow \\ X_H & \xrightarrow{\mu} & X \end{array}$$

In particular the morphisms μ_Z and $s_{H,Z} := s_Z \mu_Z$ are flat. One has [5]

- Theorem 3.5.** *i) Every toroidal embedding of G/H is of the form $X_{H,Z}$ for some $S_{\bar{H}}$ embedding Z over \mathbb{A}^ℓ . In particular it is defined and flat over \mathbf{A} .*
- ii) $X_{H,Z}$ is complete (resp. projective) if and only if the projection $Z \rightarrow \mathbb{A}^\ell$ is proper (resp. projective).*
- iii) Every G orbit in $X_{H,Z}$ is of the form $\mathcal{O}_K := (s_{H,Z})^{-1}(\mathcal{V}^* \times_{S_H} K)$ for a unique $S_{\bar{H}}$ orbit K in Z .*
- iv) Let \mathcal{F}_Z be the fan in $L_{\mathbb{R}}$ whose cones are the \tilde{W} translates of the cones in F_Z . Then \mathcal{F}_Z is the fan corresponding to S_H embedding $Z_H := (X_{H,Z})_{S_H}$. Furthermore each G orbit in $X_{H,Z}$ intersects Z_H in a unique $N_{H^\circ}(S)$ orbit (notice that in accord with iii) these orbits are in canonical bijection with $S_{\bar{H}}$ orbits in Z).*
- v) The divisors in $\Delta_{X_{H,Z}}$ are defined over \mathbf{A} .*

Proof. All these statements are proved in [5] in the case of an embedding of G/\bar{H} .

To see i) in the general case take a toroidal embedding Y of G/H . Let us take the quotient by the finite group scheme \bar{H}/H . We get an embedding of G/\bar{H} which is obviously toroidal and hence of the form X_Z for a suitable $S_{\bar{H}}$ embedding Z . If we now consider $X_{H,Z}$ we get a morphism $Y \rightarrow X_{H,Z}$ which is G equivariant birational and finite. Since both Y and $X_{H,Z}$ are normal it follows that the above morphism is a G equivariant isomorphism.

The proof of the remaining statements is now easy and we leave it to the reader. \square

Remark 3.6. 1) Let us point out that our result in particular implies that the G orbits in $X_{H,Z}$ are exactly the preimages of G orbits in X_Z .

2) It is not hard to see that $X_{H,Z}$ is smooth if and only if Z_H is smooth. Equivalently if and only if the S_H embedding whose fan is F_Z is smooth. This depends very much on the lattice $\text{Hom}(\Lambda_{S_H}, \mathbb{Z}) \subset L_{\mathbb{R}}$.

3) There exists an open affine covering $\{U_i = \text{Spec } R_i\}$ of the \mathbf{A} form of $X_{H,Z}$ such that R_i are \mathbf{A} free modules and $U_i \cap U_j = \text{Spec } R_{ij}$ where R_{ij} is also a \mathbf{A} free module.

3.3. Line bundles on a toroidal embedding. In this section we assume Y to be a smooth toroidal compactification of G/H with the \mathbf{A} structure described in the previous section.

We have the following Lemma about the structure of the Picard group of Y .

Lemma 3.7. *Let Y be a equivariant smooth toroidal compactification of G/H then*

i) We have the following sequence describing the Picard group of Y :

$$0 \longrightarrow \bigoplus_{D \in \Delta_Y} \mathbb{Z}\mathcal{O}(D) \longrightarrow \text{Pic}(Y) \xrightarrow{j_Y^*} \Lambda_H \longrightarrow 0.$$

ii) $\text{SPic}_0(Y) = \bigoplus_{D \in \Delta_Y} \mathbb{Z}\mathcal{O}(D)$.

- iii) For each closed G orbit O of Y consider the restriction $i_O^* : \mathrm{SPic}_0(Y) \longrightarrow \mathrm{Pic}(O)$ of line bundles to O . Then the product of these restriction maps $i^* : \mathrm{SPic}_0(Y) \longrightarrow \prod \mathrm{Pic}(O)$ is injective.
- iv) All line bundles on Y are defined and flat over \mathbf{A} .

Proof. The only thing we need to show to prove i) is the injectivity of the map from $\bigoplus_{D \in \Delta_Y} \mathbb{Z}\mathcal{O}(D)$ to $\mathrm{Pic}(Y)$. Notice that, since G is semisimple and simply connected and $\mathrm{Pic}(Y)$ discrete, every line bundle has a unique G linearization. Thus $\mathrm{Pic}(Y) \simeq \mathrm{Pic}_G(Y)$. It follows that it is enough to prove the injectivity of the map from $\bigoplus_{D \in \Delta_Y} \mathbb{Z}\mathcal{O}(D)$ to $\mathrm{Pic}_G(Y)$. Consider the restriction map $\mathrm{Pic}_G(Y) \rightarrow \mathrm{Pic}_{S_H}(Y_S^+)$. $\mathrm{Pic}_{S_H}(Y_S^+)$ is isomorphic to $\bigoplus \mathbb{Z}\mathcal{O}(D')$ where the sum is take over all S_H equivariant divisors D' . So the claim follows from Theorem 3.5. Since SPic_0 is the kernel of j_Y^* this also proves ii).

iii) follows from the previous considerations and the fact that, up to isomorphism a S_H equivariant line bundle on Y_S^+ is completely determined by its restriction to the closed orbits, that is the S_H fixpoints in Y_S^+ .

Finally let $D \in \Delta_Y$. By Theorem 3.5 v) it is defined over \mathbf{A} . We know that $\Lambda_H = \mathrm{Pic}(G/H)$ is generated by the codimension 1 irreducible B orbits in G/H and that these orbits are defined over \mathbf{A} by Lemma 2.7 in [6] and Lemma 2.1. Thus iv) follows from i). \square

Let F_Y be the fan associated to the toric variety Y_S^+ and let $F_Y(i)$ be the set of faces of F_Y of dimension i . In particular the closed orbits of Y are parametrized by $F_Y(\ell)$ while $F_Y(1)$ can be identified with Δ_Y the set of G invariant divisors. For each $\rho \in F_Y$ we set $y_\rho := y_\eta$ for η a generic element in ρ and denote by $O_\rho = G y_\rho$ the associated G orbit.

By Theorem 3.5 and the description of the equivariant Picard group of a toric variety we have the following description of the strictly spherical line bundles on Y :

$$(3) \quad \mathrm{SPic}_0(Y) = \{ \underline{\lambda} = (\lambda_\tau) \in \prod_{\tau \in F_Y(\ell)} \Omega_H : \lambda_\tau = \lambda_{\tau'} \text{ on } \tau \cap \tau' \}.$$

We can think of $\underline{\lambda}$ as a real valued function on the Weyl chamber C which coincides with the linear form λ_τ on the face τ . We denote by $\mathcal{L}_{\underline{\lambda}}$ a line bundle whose class is given by $\underline{\lambda}$. In particular we can describe in this way the line bundles $\mathcal{O}(D)$ for each divisor $D \in \Delta_Y$. Indeed let $v_D \in \Lambda_{S_H}^\vee$ be a non divisible element of Λ_{S_H} in the 1 dimensional face of F_Y associated to D . For each $\tau \in F_Y(\ell)$ notice that, since Y is smooth, the set $\{v_{D'} : D' \in \Delta_Y\} \cap \tau$ is a basis of $\Lambda_{S_H}^\vee$.

So we can define $\alpha_{D,\tau} \in \Lambda_{S_H}$ to be the weight which is equal to zero if $v_D \notin \tau$ while if $v_D \in \tau$ it is 1 on v_D and zero on each $v_{D'} \in \tau$ with $D' \neq D$. It is then easy to see that $\underline{\alpha}_D = (\alpha_{D,\tau})_{\tau \in F_Y(\ell)}$ is the class of $\mathcal{O}(D)$ in $\mathrm{SPic}_0(Y)$.

Now we want to describe the sections of a strictly spherical line bundle on Y in the case of characteristic 0. The proofs are very similar to the one given in [1]. A description of the section of a line bundle on a general spherical variety is given in [2] and we could have used that result as well. However the description we are going to give is more suited to our purpose.

For $D \in \Delta_Y$ let s_D be a G invariant section of $\Gamma(Y, \mathcal{O}(D))$ vanishing on D . If $\underline{\lambda} = \sum_D a_D \underline{\alpha}_D$ we set $s^{\underline{\lambda}} = \prod_D s_D^{a_D}$. Also for a given $\mu \in \Omega_H^+$ consider the line bundle $\phi^*(\mathcal{L}_\mu)$ where $\phi : Y \rightarrow X$ is the G equivariant projection from Y to X . This line bundle corresponds to the element $\underline{\mu} \in \text{SPic}_0(Y)$ with $\mu_\tau = \mu$ for all $\tau \in F_Y(\ell)$ under the identification of Ω_H with Λ_{S_H} given by Lemma 2.5 v). In particular V_μ^* is a submodule of $\Gamma(Y, \phi^*(\mathcal{L}_\mu))$.

For $\underline{\lambda} \in \text{SPic}_0(Y)$ set

$$\begin{aligned} \mathcal{A}(\underline{\lambda}) &= \{\mu \in \Omega_H^+ : \forall \tau \in F_Y(\ell), \lambda_\tau - \mu = \sum_{v_D \in \tau} a_D \underline{\alpha}_D \text{ with } a_D \geq 0\} \\ &= \{\mu \in \Omega_H^+ : \mu \leq \underline{\lambda} \text{ on } C\} \end{aligned}$$

We then have the following Theorem whose proof is completely analogous to the one given in [1].

Theorem 3.8. *Assume Y to be a smooth toroidal compactification of G/H and assume the field to be of characteristic zero and let $\underline{\lambda} \in \text{SPic}_0(Y)$ then*

$$\Gamma(Y, \mathcal{L}_{\underline{\lambda}}) = \bigoplus_{\mu \in \mathcal{A}(\underline{\lambda})} s^{\underline{\lambda} - \mu} V_\mu^*.$$

From the above result we can deduce, as in [1] §4.2 and [15] the following Corollary. Let Ω_H^{++} be the set of elements of Ω_H that are in the interior of the Weyl chamber C .

Corollary 3.9. *Let Y be a smooth toroidal compactification of G/H , let $\underline{\lambda} \in \text{SPic}_0(Y)$ and assume the field to be of characteristic zero. Then*

- i) *For every $\mu \in \Lambda^+$ V_μ^* is an irreducible summand of $\Gamma(Y, \mathcal{L})$ if and only if $\Gamma(Y_S, \mathcal{L}|_{Y_S})$ has a section of S_H weight equal to μ .*
- ii) *For every line bundle \mathcal{L} generated by global sections, the restriction map*

$$\Gamma(Y, \mathcal{L}) \rightarrow \Gamma(Y_S, \mathcal{L}|_{Y_S})$$

is surjective.

- iii) *$\mathcal{L}_{\underline{\lambda}}$ is an ample line bundle if and only if it is very ample.*
- iv) *$\mathcal{L}_{\underline{\lambda}}$ is an ample line bundle if and only if $\lambda_\tau \in \Omega_H^{++}$ and $\lambda_\tau < \lambda_{\tau'}$ on $\tau' \setminus \tau$ for all faces τ and τ' of F_Y of maximal dimension.*

Remark 3.10. We have limited our discussion to strictly spherical line bundle and to characteristic 0. Using Frobenius splitting methods it is easy to generalize the previous results as in [6]. However the stated result are enough for our purpose here.

4. THE QUOTIENT OF A SYMMETRIC VARIETY

Let Y be an embedding of G/H and let K be a subgroup such that $H^o \subset K \subset H$. Any line bundle on Y has a G linearization, so in particular it has a K linearization. Recall that if \mathcal{L} is an ample line bundle on Y a point y on Y is said to be \mathcal{L} semistable (with respect to the action of K) if for some $n > 0$ there exists $f \in \Gamma(Y, \mathcal{L}^n)^K$ such that $f(y) \neq 0$. We denote by $Y^{ss}(\mathcal{L})$ the set of \mathcal{L} semistable points, or in case \mathcal{L} is chosen just semistable points. $Y^{ss}(\mathcal{L})$ is a possibly empty open subset of Y . By [13] Theorem 3.21 there exists a good quotient of the set of \mathcal{L} semistable points which we shall denote by $K \backslash \backslash_{\mathcal{L}} Y$.

In this section we are going to prove the following Theorem

Theorem 4.1. *Let Y be an equivariant projective embedding of G/H and let \mathcal{L} be an ample and spherical line bundle on Y then the inclusion $Y_S \subset Y$ induces an isomorphism of algebraic varieties between $\tilde{W} \backslash Y_S$ and $K \backslash \backslash_{\mathcal{L}} Y$.*

We will prove this Theorem by computing the invariant sections. We will analyze first the case of the wonderful compactification and the case of the quotient of the open affine part G/H .

4.1. Invariants and semiinvariants of the Cox ring of the wonderful compactification. In this section we compute the H invariants of the ring $R(X)$. We use the notations introduced in section 3.1.

Lemma 4.2. *Let $\lambda \in \Pi$. Then $(\text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H = \text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)^H)$. In particular the dimension of the space of invariants $\Gamma(X, \mathcal{L}_\lambda)^H$ equals the cardinality of the set $K_\lambda := \{\mu \in \Omega^+ : \mu \leq_\sigma \lambda\}$ if $\lambda \in \Omega_H$ and is zero otherwise.*

Proof. In characteristic zero the equality

$$(\text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H = \text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)^H)$$

is an immediate consequence of the linear reductivity of H .

Also (in arbitrary characteristic) by equation (1) we have that

$$(\text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H = \bigoplus_{\mu \in \Pi^+ \text{ and } \mu \leq_\sigma \lambda} s^{\lambda - \mu} (V_\mu^*)^H.$$

By Vust criterion (Theorem 2.3) $(V_\mu^*)^H$ is one dimensional if $\mu \in \Omega_H^+$ and it is zero otherwise. So, since by Lemma 2.5 $\mathbb{Z}[\tilde{\Phi}] \subset \Omega_H$ we have that $(\text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H$ has dimension equal to $|K_\lambda|$ if $\lambda \in \Omega_H$ and it is zero otherwise.

In general $(\text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H \supset \text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)^H)$ so

$$\dim \Gamma(X, \mathcal{L}_\lambda)^H \leq \dim (\text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H.$$

On the other hand by Theorem 3.1 and Lemma 3.7 the variety X and the spaces $\Gamma(X, \mathcal{L}_\lambda)$ are all defined over \mathbf{A} . Lemma 2.2 then clearly implies that in positive characteristic $\dim \Gamma(X, \mathcal{L}_\lambda)^H$ can only increase. This together with the previous inequality implies our claim. \square

We compute now the ring $R(X)^H$. By Lemma 4.2, for each $\tilde{\alpha} \in \tilde{\Delta}$ we can choose $p_{\tilde{\alpha}} \in \Gamma(X, \mathcal{L}_{\tilde{\omega}_{\tilde{\alpha}}})$ an H^o invariant section which does not vanish on $X_{\tilde{\Delta}}$. So, if $\lambda = \sum a_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} \in \Omega^+$, we can define

$$(4) \quad p^\lambda = \prod_{\tilde{\alpha} \in \tilde{\Delta}} p_{\tilde{\alpha}}^{a_{\tilde{\alpha}}}.$$

Proposition 4.3. *The set $\{s^\mu p^\lambda : \mu \in \Pi \text{ and } \mu \geq_\sigma 0 \text{ and } \lambda \in \Omega_H^+\}$ is a \mathbb{k} basis of $R(X)^H$. In particular the ring $R(X)^{H^o}$ is a polynomial ring in the variables $s_{\tilde{\alpha}}, p_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \tilde{\Delta}$.*

Proof. Notice first that by Lemma 4.2, if $\lambda \in \Omega_H$, $\Gamma(X, \mathcal{L}_\lambda)^H = \Gamma(X, \mathcal{L}_\lambda)^{H^o}$ so it is enough to prove the claim in the case of H^o .

The image of $p_{\tilde{\alpha}}$ in the graded ring $\text{Gr}_F(R(X))$ defines an H^o invariant element $\bar{p}_{\tilde{\alpha}}$ of $V_{\tilde{\omega}_{\tilde{\alpha}}}^*$. So by the description of the H^o invariants of $\text{Gr}_F(R(X))$ the image of the elements $s^\mu p^\lambda$ in the graded ring $\text{Gr}_F(R(X))$ is a \mathbb{k} basis of the space of H^o invariants. This implies that the elements $s^\mu p^\lambda$ are linearly independent.

By construction, the elements $s^\mu p^\lambda$ are H^o invariants. So, again by Lemma 4.2 they are a \mathbb{k} basis of $R(X)^{H^o}$. \square

The computation of semi invariants is similar. If V is a representation of \bar{H} we denote by $V_{si}^{\bar{H}}$ the subspace spanned by the set of semi invariant vectors i.e. vectors spanning lines fixed by \bar{H} .

By Theorem 2.4, we know that there are semi invariants which are not H^o invariants only if there exists an exceptional simple root. Set $\Delta_e = \{\alpha \in \Delta_1 : \alpha \text{ is exceptional}\}$ and $\tilde{\Delta}_{ne} = \{\tilde{\alpha} \in \tilde{\Delta} : \alpha \text{ is not exceptional}\}$. By Theorem 2.4 the set $\{\omega_\alpha : \alpha \in \Delta_e\} \cup \{\tilde{\omega}_{\tilde{\alpha}} : \tilde{\alpha} \in \tilde{\Delta}_{ne}\}$ is a basis of Π . Let $\bar{q}_\alpha \in V_{\omega_\alpha}^*$ be a non zero \bar{H} semi invariant. \bar{q}_α is unique up to multiplication by a non zero scalar. So the ring $(R_\Pi(X_{\tilde{\Delta}}))_{si}^{\bar{H}}$ of semi invariants is a polynomial ring in the generators \bar{q}_α with $\alpha \in \Delta_e$ and $\bar{p}_{\tilde{\alpha}}$ (the restriction of $p_{\tilde{\alpha}}$ to $X_{\tilde{\Delta}}$) with $\tilde{\alpha} \in \tilde{\Delta}_{ne}$. Using Corollary 2.6 and arguing as in Lemma 4.2 we deduce that there exists $q_\alpha \in \Gamma(X, \mathcal{L}_{\omega_\alpha})_{si}^{\bar{H}}$ such that its restriction to $X_{\tilde{\Delta}}$ is equal to \bar{q}_α . If $\lambda = \sum_{\alpha \in \Delta_e} c_\alpha \omega_\alpha + \sum_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} \in \Pi^+$, we define $q^\lambda = \prod_{\alpha \in \Delta_e} q_\alpha^{c_\alpha} \cdot \prod_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} p_{\tilde{\alpha}}^{c_{\tilde{\alpha}}}$. The arguments given in the case of invariants can now be easily adapted implying

Proposition 4.4. *Let $\lambda \in \Pi$. Then $(\text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))_{si}^{\bar{H}} = \text{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)_{si}^{\bar{H}})$. Moreover the set $\{s^\mu q^\lambda : \mu \in \Pi \text{ and } \mu \geq_\sigma 0 \text{ and } \lambda \in \Pi^+\}$ is a \mathbb{k} basis of $R(X)_{si}^{\bar{H}}$. In particular the ring $R(X)_{si}^{\bar{H}}$ is a polynomial ring in the variables $s_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \tilde{\Delta}$, $p_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \tilde{\Delta}_{ne}$ and q_α with $\alpha \in \Delta_e$.*

4.2. Filtration of the coordinate ring of G/H and Richardson Theorem. We want now to use the wonderful variety X to define a filtration of the coordinate ring of G/H . In the case in which H is the diagonal subgroup in $G = H \times H$, these ideas already appear in [18]. This will be used

to describe the H invariants of this ring. This description of the invariants has already been given by Richardson [14] but a proof in our setting seems natural.

First we make explicit the relation between the coordinate ring of G/H and the ring $R_{\Omega_H}(X)$.

For each $\tilde{\alpha} \in \tilde{\Delta}$ we choose a trivialization $\varphi_{\tilde{\omega}_{\tilde{\alpha}}} : \pi^*(\mathcal{L}_{\tilde{\omega}_{\tilde{\alpha}}}) \longrightarrow \mathcal{O}_{G/H^\circ}$. Given $\lambda = \sum_{\tilde{\alpha} \in \tilde{\Delta}} c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} \in \Omega$ we obtain the trivialization of $\pi^*(\mathcal{L}_\lambda)$ given by $\otimes_{\tilde{\alpha} \in \tilde{\Delta}} \varphi_{\tilde{\omega}_{\tilde{\alpha}}}^{\otimes c_{\tilde{\alpha}}}$. With these choices the pull back of sections defines a ring homomorphism:

$$\pi_H^* : R_{\Omega_H}(X) \longrightarrow \mathbb{k}[G/H].$$

Notice also that since $s_{\tilde{\alpha}}$ is G invariant the functions $\pi_H^*(s_{\tilde{\alpha}})$ are constant and we can normalize them to be equal to 1.

For each $\lambda \in \Omega_H$ we consider the G submodule

$$\mathbb{F}_\lambda := \pi_H^*(\Gamma(X, \mathcal{L}_\lambda))$$

of $\mathbb{k}[G/H]$. Notice that since the $s_{\tilde{\alpha}}$ are all equal to 1, we clearly have that if $\mu <_\sigma \lambda$, $\mathbb{F}_\mu \subset \mathbb{F}_\lambda$. Also, since the image of π_H is dense in X we have that π_H^* restricted to $\Gamma(X, \mathcal{L}_\lambda)$ is an isomorphism onto \mathbb{F}_λ . Furthermore if we set $\mathbb{F}'_\lambda = \sum_{\mu <_\sigma \lambda} \mathbb{F}_\mu$, we have $\mathbb{F}_\lambda / \mathbb{F}'_\lambda \simeq V_\lambda^*$.

Proposition 4.5. *The map π_H^* induces an isomorphism of rings*

$$\varphi : \frac{R_{\Omega_H}(X)}{(s_{\tilde{\alpha}} - 1 : \tilde{\alpha} \in \tilde{\Delta})} \longrightarrow \mathbb{k}[G/H].$$

Proof. φ is clearly well defined and its surjectivity follows immediately from Proposition 2.8.

Let us now show that φ is injective. As above set $\Xi_H = \Omega_H / \Omega_{\bar{H}}$ and for all cosets $\xi \in \Xi_H$ define $R_\xi = \bigoplus_{\lambda \in \xi} \Gamma(X, \mathcal{L}_\lambda)$ so that $R_{\Omega_H}(X) = \bigoplus_{\xi \in \Xi_H} R_\xi$ is a Ξ_H grading of the ring $R_{\Omega_H}(X)$.

On the other hand by Proposition 2.8, the coordinate ring $\mathbb{k}[G/H]$ decomposes as the direct sum $\bigoplus_{\xi \in \Xi_H} \Gamma(G/\bar{H}, \mathcal{L}_\xi)$ and the restriction of π_H^* decomposes as the direct sum $\bigoplus_{\xi \in \Xi_H} j_\xi^*$ where $j_\xi^* : R_\xi \rightarrow \Gamma(G/\bar{H}, \mathcal{L}_\xi)$ is induced by the inclusion j_X of G/\bar{H} in X .

Also, since the elements $\{s_{\tilde{\alpha}} - 1 : \tilde{\alpha} \in \tilde{\Delta}\}$ lie in R_0 the ideal I that they generate decomposes as the direct sum $I = \bigoplus_{\xi \in \Xi_H} I_\xi$ with $I_\xi = I \cap R_\xi$, for each $\xi \in \Xi_H$. Thus j_ξ^* induces a map

$$\varphi_\xi : R_\xi / I_\xi \rightarrow \Gamma(G/\bar{H}, \mathcal{L}_\xi)$$

and it is enough to see that φ_ξ is injective for each $\xi \in \Xi_H$.

Fix $\xi \in \Xi_H$. Let $g = \sum_{\lambda \in A} g_\lambda \in R_\xi$ with $g_\lambda \in \Gamma(X, \mathcal{L}_\lambda)$ and A a finite subset of the coset ξ . Assume $\pi_H^*(g) = 0$. By assumption there exists $\mu \in \xi$ such that $\mu \geq_\sigma \lambda$ for all $\lambda \in A$. Set $g' = \sum_{\lambda \in A} s^{\mu-\lambda} g_\lambda$ and notice that $g' \equiv g \pmod{I_\xi}$ and that $g' \in \Gamma(X, \mathcal{L}_\mu)$. We have $\pi_H^*(g') = \pi_H^*(g) = 0$ and since π_H^* restricted to $\Gamma(X, \mathcal{L}_\mu)$ is injective, $g' = 0$ and $g \in I_\xi$ as desired. \square

Corollary 4.6. *The G submodules \mathbb{F}_λ , $\lambda \in \Omega_H$, induce a good (increasing) filtration of the coordinate ring $\mathbb{k}[G/H]$.*

We are now going to use this filtration to study the ring of invariants $\mathbb{k}[G/H]^K$. We first need a well known lemma.

Lemma 4.7. *Fix a dominant weight $\lambda \in \Omega^+$.*

- i) Let $\phi \in V_\lambda^*$ be a nonzero H^o invariant and let us consider the decomposition of V_λ^* with respect to the action of T . Then the lowest weight component of ϕ is not zero;*
- ii) Let $p^\lambda \in \Gamma(X, \mathcal{L}_\lambda)$ denote the H^o invariant defined in formula (4) and consider the decomposition of $p^\lambda \in \Gamma(X, \mathcal{L}_\lambda)$ with respect to the action of T . Then the lowest weight component of p^λ is not zero.*

Proof. In V_λ^* there is a non zero vector v fixed by the maximal unipotent subgroup U^- opposite to B . This vector is unique up to a non zero scalar and has weight $-\lambda$ which is the lowest weight of V_λ^* . Write $\phi = w + av$ with w lying in the unique T stable complement V' of the one dimensional space spanned by v and $a \in \mathbb{k}$. V' is B stable. We need to prove $a \neq 0$.

Consider the G submodule W generated by ϕ . Since W must contain a non zero vector fixed by U^- , it has to contain v .

On the other hand $B \cdot H^o$ is dense in G so the subspace W is equal to the space spanned by the vectors $b \cdot \phi$ with $b \in B$. If $a = 0$ then W would be contained in V' giving a contradiction. This proves *i*).

To see *ii*) it suffices to consider the image \bar{p}^λ in $\Gamma(X, \mathcal{L}_\lambda)/F'_{\lambda, \lambda} \simeq V_\lambda^*$ which is non zero by the very definition of p^λ . \square

We can now prove Richardson Theorem (see [14] Corollary 11.5). Notice that, since $N_{H^o}(S) \subset H^o \subset K$ the inclusion of S_H in G/H induces a map from $\tilde{W} \backslash S_H$ to $K \backslash G/H$.

Theorem 4.8. *Let $H^o \subset K \subset H$ be a subgroup of H . Then the inclusion $S_H \subset G/H$ induces an isomorphism $\tilde{W} \backslash S_H \simeq K \backslash G/H$.*

Proof. By the definition of \tilde{W} , the restriction of functions from G/H to S_H induces a homomorphism

$$d : \mathbb{k}[G/H]^K \rightarrow \mathbb{k}[S_H]^{\tilde{W}}.$$

We claim that d is an isomorphism.

To see this we first make some remarks on the K invariants of $\mathbb{k}[G/H]$. For $\lambda \in \Omega_H^+$ let $f^\lambda := \pi_H^*(p^\lambda)$. Arguing as in Proposition 4.3 it is easy to see that the elements f^λ with $\lambda \in \Omega_H^+$ are a basis of $\mathbb{k}[G/H]^K$ as a vector space. In particular for each $\lambda \in \Omega_H^+$, $\mathbb{F}_\lambda^K / \mathbb{F}'_\lambda^K$ is one dimensional and spanned by the class of f^λ (notice that $\Omega_H \subset \Omega_K$).

The computation of the \tilde{W} invariants of the ring $\mathbb{k}[S_H]$ is also very simple. Let $\mathbb{k}[S_H] = \bigoplus_{\lambda \in \Lambda_{S_H}} \mathbb{k}\varphi_\lambda$ where φ_λ is a function of weight λ . We know by Theorem 2.3 and Lemma 2.5 that the restriction r of character from T to

S induces an isomorphism between Λ_{S_H} and Ω_H so we identify the two lattices. Also a weight $\lambda \in \Omega_H$ is dominant with respect to $\tilde{\Delta}$ if and only if it is dominant with respect to Δ . If $\lambda \in \Lambda_{S_H}^+$ we set

$$\psi_\lambda = \sum_{\psi \in \tilde{W} \cdot \varphi_{-\lambda}} \psi.$$

The elements ψ_λ with $\lambda \in \Omega_H^+$ are clearly a basis of $\mathbb{k}[S_H]^{\tilde{W}}$.

Given $\lambda \in \Omega_H^+$, let U_λ denote the span of elements ψ_μ with $\mu \in \Omega_H^+$ and $\mu \leq_\sigma \lambda$.

Notice that $f_S^\lambda := d(f^\lambda)$ lies in U_λ . Indeed f_S^λ is a \tilde{W} invariant and its weights are in a subset of the weights appearing in V_λ^* . Thus for each $\lambda \in \Omega_H^+$, $d(\mathbb{F}_\lambda)^K \subseteq U_\lambda$. We claim that d maps isomorphically \mathbb{F}_λ^K onto U_λ . This will imply our claim.

By an easy induction we need to show that $f_S^\lambda \notin \sum_{\mu <_\sigma \lambda \text{ and } \mu \in \Omega_H^+} U_\mu$. Using Lemma 4.7 it suffices to prove that the restriction to S_H of a lowest weight vector h in \mathbb{F}_λ is non zero. The closure of S_H in X contains the unique point of the closed orbit $X_{\tilde{\Delta}}$ fixed by B . h does not vanish at this point. Since h is non zero at a point in the closure of S_H it cannot vanish on S_H proving that f_S^λ does not lie in $\sum_{\mu <_\sigma \lambda \text{ and } \mu \in \Omega_H^+} U_\mu$. \square

Remark 4.9. Notice that in particular $K \backslash G/H$ does not depend on the choice of the subgroup K between H^o and H . However it is not true in general that the K orbits in G/H are the same of the H orbits in G/K . To see this it is enough to take $G = SL(2, \mathbb{C})$ and σ the conjugation by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then H^o is the diagonal torus and it is easy to check that $H^o \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bar{H} \neq \bar{H} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bar{H}$.

To complete our picture we show, with a different proof, another result of Richardson which tells us the orbits of the elements in S_H are precisely the closed orbits in G/H .

Proposition 4.10. *Let $H^o \subset H, K \subset \bar{H}$. Let $s \in S_H$ then Ks is closed in G/H .*

Proof. Since H^o has finite index both in H and in K , it is enough to study the case $H = K = H^o$.

Fix V to be a finite dimensional faithful representation of G . Consider the map $\chi : G/H^o \rightarrow G \subset GL(V)$ given by $\chi(gH^o) = g\sigma(g)^{-1}$. By [16] Theorem 5.4.4 this is a closed immersion. Notice that $\chi(h \cdot x) = h\chi(x)h^{-1}$ for all $h \in H^o$ and $x \in G/H^o$.

For $s \in S_H$ set $x = \chi(s) = s^2$. The element s is semisimple in $GL(V)$. We want to prove that the orbit $\{h x h^{-1} : h \in H^o\}$ is closed in $GL(V)$.

Let $p(t)$ be the minimal polynomial of x . Since x is semisimple $p(t)$ does not have multiple roots. For all $\lambda \in \mathbb{k}^*$ and $y \in G$ set $V_\lambda(y) = \{v \in \mathfrak{g} :$

$Ad_y(v) = \lambda v\}$. Notice that for $y \in T$

$$V_\lambda(y) = \begin{cases} \bigoplus_{\alpha \in \Phi: \alpha(y)=\lambda} \mathfrak{g}_\alpha & \text{if } \lambda \neq 1 \\ \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi: \alpha(y)=1} \mathfrak{g}_\alpha & \text{if } \lambda = 1. \end{cases}$$

Given $\lambda \in \mathbb{k}^*$ and $\mu = \frac{1}{2}(\lambda + \lambda^{-1})$ set

$$W_\mu(y) = \begin{cases} V_\lambda(y) \oplus V_{\lambda^{-1}}(y) & \text{if } \lambda \neq \pm 1 \\ V_\lambda(y) & \text{if } \lambda = \pm 1. \end{cases}$$

Notice that we have

$$(5) \quad \mathfrak{h} = \bigoplus_{\mu} (\mathfrak{h} \cap W_\mu(x))$$

since $\mathfrak{h} = \mathfrak{t}^\sigma \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi_1^+} \mathbb{k}(x_\alpha + \sigma(x_\alpha))$ and $\sigma(\alpha)(x) = \alpha(x)^{-1}$ so that $x_\alpha + \sigma(x_\alpha) \in W_{\alpha(x) + \alpha(x)^{-1}}$. Let $n_\mu = \dim(\mathfrak{h} \cap W_\mu(x))$. We define the closed subset $M \subset G$ as follows. An element $y \in G$ lies in M if

- i) $p(y) = 0$
- ii) the characteristic polynomial of Ad_y is equal to the characteristic polynomial of Ad_x
- iii) $\dim(\mathfrak{h} \cap W_\mu(y)) \geq n_\mu$ for all μ .

By condition iii) and relation (5) it follows that if $y \in M$, $\dim(\mathfrak{h} \cap W_\mu(y)) = n_\mu$ for all μ . In particular $\dim \mathfrak{h} \cap V_1(y) = \dim \mathfrak{h} \cap V_1(x)$ so that $\dim \mathfrak{h} \cap Z_{\mathfrak{g}}(y) = \dim \mathfrak{h} \cap Z_{\mathfrak{g}}(x)$.

Now by condition i) if $y \in M$, y is semisimple and so by [16] Theorem 5.4.4 we have that $Z_{\mathfrak{g}}(y)$ is the Lie algebra of $Z_G(y)$. It follows that $\dim \mathfrak{h} \cap Z_{\mathfrak{g}}(y) = \dim H \cap Z_G(y) = n_1$ and finally $\dim H^o y = \dim H - n_1$ so that every H^o orbit in M has the same dimension.

In particular every H^o orbit in M is closed. Since $x \in M$ the Proposition follows. \square

Remark 4.11. Notice that our proof works also under the slightly more general assumption that G is reductive. This will be useful later on.

Remark 4.12. We notice that if $H^o \subset K \subset H$ and $s \in S$ then $Ks = Hs$ in G/H . Indeed by Remark 4.9 we have that $K \backslash G/H = H \backslash G/H$ so the natural map $Kx \mapsto Hx$ from the set of K orbits into the set of H orbits is a bijection at the level of closed orbits. At this point everything follows from the fact that Hs is a union of closed K orbits.

4.3. The quotient of a smooth projective toroidal compactification.

Let us now assume that \mathcal{L} is a line bundle generated by global sections but not necessarily ample on a given smooth, projective toroidal compactification Y of G/H . This is going to be useful in order to prove Theorem 4.1 in its full generality.

Since \mathcal{L} is spherical, $\pi_Y^*(\mathcal{L})$ is trivial. Notice now that the G linearization of \mathcal{L} restricts to a $N_{H^o}(S)$ linearization of $\mathcal{L}|_{Y_S}$ and that $Z_{H^o}(S)$ acts trivially

on $\mathcal{L}|_{Y_S}$. To see this first notice that $Z_{H^o}(S)$ acts trivially on the fiber of \mathcal{L} over y_0 . Indeed $\pi_Y^*(\mathcal{L})$ is trivial, hence H^o acts trivially on the fiber of \mathcal{L} over y_0 and $Z_{H^o}(S) \subset H^o$. Now, $Z_{H^o}(S)$ commutes with S and $S \cdot y_0$ is dense in Y_S so necessarily $Z_{H^o}(S)$ acts trivially on $\mathcal{L}|_{Y_S}$.

Notice that $Y^{ss}(\mathcal{L}) = Y^{ss}(\mathcal{L}^n)$ for any $n > 0$, so $K \backslash \backslash_{\mathcal{L}^n} Y \simeq K \backslash \backslash_{\mathcal{L}} Y$. Since Ω/Ω_K is finite, up to taking a power of \mathcal{L} we can always assume that $\pi_{Y,K}^*(\mathcal{L}) \simeq \mathcal{O}_{G/K}$ where $\pi_{Y,K} : G/K \rightarrow Y$ is defined by $\pi_{Y,K}(gK) = g \cdot y_0$. We call these line bundles K spherical. We have

Theorem 4.13. *Let \mathcal{L} be a K spherical line bundle on Y . Then*

$$\Gamma(Y, \mathcal{L})^K \simeq \Gamma(Y_S, \mathcal{L}|_{Y_S})^{\bar{W}}.$$

We need first a well known general fact on flat schemes over \mathbf{A} whose proof we give for completeness.

Lemma 4.14. *Let U be a projective flat scheme over \mathbf{A} such that there exists an open affine covering $\{U_1, \dots, U_n\}$ of U with $U_i = \text{Spec } R_i$ and $U_i \cap U_j = \text{Spec } R_{ij}$ where R_i and R_{ij} are free \mathbf{A} modules. Let \mathcal{L} be a locally free sheaf over U . For each ring extension $\mathbf{A} \rightarrow B$ let $U_B = U \times_{\text{Spec}(\mathbf{A})} \text{Spec}(B)$ and \mathcal{L}_B the pull back of \mathcal{L} to U_B . Then:*

- i) $\Gamma(U, \mathcal{L})$ is a finitely generated free \mathbf{A} module;
- ii) the map $B \otimes_{\mathbf{A}} \Gamma(U, \mathcal{L}) \rightarrow \Gamma(U_B, \mathcal{L}_B)$ is injective;
- iii) moreover if B is a field of characteristic 0 then we have an isomorphism $B \otimes_{\mathbf{A}} \Gamma(U, \mathcal{L}) \simeq \Gamma(U_B, \mathcal{L}_B)$.

Proof. The fact that $\Gamma(U, \mathcal{L})$ is finitely generated is the content of Theorem III.5.2 point a) in [8].

We can refine the covering U_i in such a way it has the same properties and moreover it is such that $\mathcal{L}|_{U_i}$ is defined by a free R_i module of rank 1. We have the exact sequence:

$$0 \rightarrow \Gamma(U, \mathcal{L}) \xrightarrow{r_1} \prod \Gamma(U_i, \mathcal{L}) \xrightarrow{r_2} \prod \Gamma(U_i \cap U_j, \mathcal{L})$$

where r_1 and r_2 are given by restriction of sections. In particular $\Gamma(U, \mathcal{L})$ is a submodule of $\prod \Gamma(U_i, \mathcal{L})$ which is a free \mathbf{A} module, hence, since \mathbf{A} is a PID, $\Gamma(U, \mathcal{L})$ is a free \mathbf{A} module.

Let M denote the image of r_2 . M is a submodule of $\prod \Gamma(U_i \cap U_j, \mathcal{L})$, so also M is a free \mathbf{A} module. Write $r_2 = \iota \circ r'$ with $r' : \prod \Gamma(U_i, \mathcal{L}) \rightarrow M$ and $\iota : M \rightarrow \prod \Gamma(U_i \cap U_j, \mathcal{L})$. For any \mathbf{A} algebra B we can tensor by B and, since by definition $\Gamma(U_i \times_{\text{Spec}(\mathbf{A})} \text{Spec}(B), \mathcal{L}_B) = B \otimes_{\mathbf{A}} \Gamma(U_i, \mathcal{L})$, we get the exact sequence

$$(6) \quad 0 \rightarrow \Gamma(U, \mathcal{L}) \otimes_{\mathbf{A}} B \xrightarrow{r_1 \otimes \text{id}_B} \prod \Gamma(U_i \times_{\text{Spec}(\mathbf{A})} \text{Spec}(B), \mathcal{L}_B) \xrightarrow{r' \otimes \text{id}_B} M \otimes_{\mathbf{A}} B \rightarrow 0$$

from which deduce that the map from $\Gamma(U, \mathcal{L}) \otimes_{\mathbf{A}} B$ to $\Gamma(U_B, \mathcal{L}_B)$ is injective. This proves i) and ii).

Now assume that B is a field of characteristic zero. ι is an inclusion between free \mathbf{A} modules, so since B has characteristic zero we have that $\iota \otimes \text{id}_B : M \otimes_{\mathbf{A}} B \longrightarrow \prod \Gamma(U_i \cap U_j, \mathcal{L}) \otimes_{\mathbf{A}} B = \prod \Gamma((U_i \cap U_j) \times_{\text{Spec}(\mathbf{A})} \text{Spec}(B), \mathcal{L}_B)$ is injective. It follows that $M \otimes_{\mathbf{A}} B$ is the image of $r_2 \otimes \text{id}_B$ and by the exact sequence (6) we get that $\Gamma(U, \mathcal{L}) \otimes_{\mathbf{A}} B$ is equal to the space of sections of \mathcal{L}_B on U_B . \square

Notice that by Remark 3.6 3) we can apply this Lemma to a toroidal compactification. We obtain

Lemma 4.15. *Let Y be a smooth toroidal compactification of G/H and let \mathcal{L} be a K spherical line bundle on Y generated by global sections. Then the restriction of sections from Y to Y_S induces an isomorphism $\Gamma(Y, \mathcal{L})^K \simeq \Gamma(Y_S, \mathcal{L}|_{Y_S})^{\bar{W}}$.*

Proof. We prove first that this map is injective. Consider the pull back $\pi_{Y,K}^*(\mathcal{L})$. By hypothesis this is isomorphic to the trivial line bundle on G/K . So a trivialization of it induces inclusions $\Gamma(Y, \mathcal{L}) \subset \mathbb{k}[G/K]$ and $\Gamma(Y_S, \mathcal{L}|_{Y_S}) \subset \mathbb{k}[S_K]$. We get a commutative diagram

$$\begin{array}{ccc} \Gamma(Y, \mathcal{L})^K & \longrightarrow & \mathbb{k}[G/K]^K \\ \downarrow & & \downarrow \\ \Gamma(Y_S, \mathcal{L}|_{Y_S})^{\bar{W}} & \longrightarrow & \mathbb{k}[S_K]^{\bar{W}} \end{array}$$

where the horizontal maps are the induced by the pull back of sections and vertical maps are given by restriction of sections. Since the inclusions $G/H \subset Y$ and $S_H \subset Y_S$ are open the two horizontal maps are injective and by Theorem 4.8 also the right vertical map is injective. It follows that also the vertical map on the left is injective.

In order to prove surjectivity it is enough to prove that we have enough invariants. First we prove this result in characteristic zero. Let U be a G submodule of $\mathbb{C}[G/K]$ and U_S be its image in $\mathbb{C}[S_K]$. Observe that by Theorem 4.8 we have an isomorphism between U^K and $U_S^{\bar{W}}$.

Set U equal to the image of $\Gamma(Y, \mathcal{L})$ in $\mathbb{k}[G/K]$. By Corollary 3.9 ii), the restriction map $\Gamma(Y, \mathcal{L}) \longrightarrow \Gamma(Y_S, \mathcal{L}|_{Y_S})$ is surjective for any spherical line bundle generated by global sections. Thus U_S equals the image of $\Gamma(Y_S, \mathcal{L}|_{Y_S})$ in $\mathbb{k}[S_K]$ and this implies our claim.

Assume now that the base field \mathbb{k} is of arbitrary characteristic. The description of $\Gamma(Y_S, \mathcal{L}|_{Y_S})$ as a S module does not depend on the characteristic. It follows that there is a basis of $\Gamma(Y_S, \mathcal{L}|_{Y_S})$ on which \bar{W} acts by permutations. Thus also the description of $\Gamma(Y_S, \mathcal{L}|_{Y_S})^{\bar{W}}$ and hence its dimension d does not depend on the characteristic. On the other hand by 4.14 ii) we have that $d \leq \dim \Gamma(Y_{\mathbb{k}}, \mathcal{L}_{\mathbb{k}})^K$. Since $\Gamma(Y_{\mathbb{k}}, \mathcal{L}_{\mathbb{k}})^K$ injects into $\Gamma(Y_S, \mathcal{L}|_{Y_S})^{\bar{W}}$ everything follows. \square

If we now set

$$A_{\mathcal{L}} := \oplus_n \Gamma(Y, \mathcal{L}^n) \quad \text{and} \quad B_{\mathcal{L}} := \oplus_n \Gamma(Y_S, \mathcal{L}^n|_{Y_S})$$

we deduce from the above Lemma that $\text{Proj}(A_{\mathcal{L}}^K) = \text{Proj}(B_{\mathcal{L}}^{\tilde{W}})$ for any spherical line bundle \mathcal{L} generated by global sections on a smooth toroidal projective embedding Y of G/H . In particular Theorem 4.1 follows for such a compactification.

4.4. Proof of Theorem 4.1. We now prove Theorem 4.1 for any projective embedding Y of G/H . Consider an equivariant resolution \tilde{Y} of the closure of the image of G/H in $Y \times X$. By construction this is a toroidal compactification and we have a G equivariant birational projective morphism $\phi : \tilde{Y} \rightarrow Y$. Clearly $\phi(\tilde{Y}_S) = Y_S$.

As already noticed at the beginning of section 4.3 we can assume \mathcal{L} to be K spherical. Let $\mathcal{M} = \phi^*(\mathcal{L})$ and notice that this is a K spherical line bundle on \tilde{Y} generated by global sections. Notice also that since Y is normal we have $\Gamma(\tilde{Y}, \mathcal{M}) = \Gamma(Y, \mathcal{L})$ and $A_{\mathcal{M}} = A_{\mathcal{L}}$. We have the following commutative diagram where the horizontal maps are given by pull back of sections, and the vertical maps are given by the restriction of sections:

$$\begin{array}{ccc} \Gamma(Y, \mathcal{L})^K & \xrightarrow{\simeq} & \Gamma(\tilde{Y}, \mathcal{M})^K \\ \downarrow & & \downarrow \\ \Gamma(Y_S, \mathcal{L}|_{Y_S})^{\tilde{W}} & \longrightarrow & \Gamma(\tilde{Y}_S, \mathcal{M}|_{\tilde{Y}_S})^{\tilde{W}} \end{array}$$

Now the vertical map on the right is an isomorphism by the result obtained for a smooth toroidal compactification and the bottom map is injective, since $\tilde{Y}_S \rightarrow Y_S$ is surjective. So also the vertical map on the left is an isomorphism. So $B_{\mathcal{L}}^{\tilde{W}} \simeq A_{\mathcal{L}}^K$ and

$$K \backslash \backslash_{\mathcal{L}} Y = \text{Proj}(A_{\mathcal{L}}^K) \simeq \text{Proj}(B_{\mathcal{L}}^{\tilde{W}}) = \tilde{W} \backslash Y_S$$

as claimed. \square

5. SEMISTABLE POINTS

In this section we want to give a more geometric description of the set of semistable points. We analyze first the case of flag varieties.

5.1. Divisors of invariants in flag varieties. We give first some definitions. If $I \subset \tilde{\Delta}$ we set $\Delta_I = \Delta_0 \cup \{\alpha \in \Delta_1 : \tilde{\alpha} \in I\}$ and define Φ_I to be the subroot system of Φ spanned by Δ_I . We let P_I denote the corresponding parabolic subgroup of G and $\Lambda_I = \Lambda_{P_I}$ the set of characters of P_I . We also set $\Pi_I = \Pi \cap \Lambda_I$, $\Omega_{I,H} = \Omega_H \cap \Lambda_I$ and $\Omega_I = \Omega_{I,H^\circ}$. We have

$$\Omega_I = \bigoplus_{\tilde{\alpha} \in \tilde{\Delta} \setminus I} \mathbb{Z} \tilde{\omega}_{\tilde{\alpha}} \quad \Pi_I = \Omega_I + \bigoplus_{\alpha \in \Delta_e \setminus \Delta_I} \mathbb{Z} \omega_{\alpha}.$$

Let us describe the set of invariant and semiinvariant sections with respect to the action of H of a line bundle \mathcal{L}_λ on G/P_I . Since if $\lambda \in \Lambda_I^+$ we have that $\Gamma(G/P_I, \mathcal{L}_\lambda) \simeq V_\lambda^*$ (and is zero if λ is not dominant). In this case we can apply directly the result of Vust without introducing any further filtration.

If $\tilde{\alpha} \in \tilde{\Delta} \setminus I$ let $\bar{p}_{\tilde{\alpha}}$ be a non zero section of $\mathcal{L}_{\tilde{\omega}_{\tilde{\alpha}}}$ on G/P_I invariant under the action of H^o . Similarly if $\alpha \in \Delta_e \setminus I$ let \bar{q}_α be a non zero semiinvariant section of $\mathcal{L}_{\omega_\alpha}$ on G/P_I . If $\lambda = \sum a_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}}$ is dominant so that $a_{\tilde{\alpha}} \geq 0$ for all $\tilde{\alpha}$, we define $\bar{p}^\lambda = \prod_{\tilde{\alpha}} \bar{p}_{\tilde{\alpha}}^{a_{\tilde{\alpha}}}$ and similarly if $\lambda = \sum_{\alpha \in \Delta_e} c_\alpha \omega_\alpha + \sum_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}}$ is dominant in Π_I we define $\bar{q}^\lambda = \prod_{\alpha \in \Delta_e} \bar{q}_\alpha^{c_\alpha} \cdot \prod_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} \bar{p}_{\tilde{\alpha}}^{c_{\tilde{\alpha}}}$.

We notice that up to a non zero scalar, \bar{p}^λ is the unique H^o invariant section of \mathcal{L}_λ and that it is H invariant if and only $\lambda \in \Omega_H$. Similarly \bar{q}^λ is the unique \bar{H} semiinvariant section of \mathcal{L}_λ .

In particular if we set $R(G/P_I) = \bigoplus_{\lambda \in \Lambda_I} \Gamma(G/P_I, \mathcal{L}_\lambda)$, we have that the ring of invariants $R(G/P_I)^{H^o}$ is a polynomial ring in $\bar{p}_{\tilde{\alpha}}$ for $\tilde{\alpha} \notin I$ and the ring of semiinvariants $R(G/P_I)^{\bar{H}}_{si}$ is a polynomial ring in $\bar{p}_{\tilde{\alpha}}, \bar{q}_\beta$ for $\tilde{\alpha} \in \tilde{\Delta}_{ne} \setminus I$ and $\beta \in \Delta_e \setminus \Delta_I$.

If $\lambda = \sum c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} \in \Omega$ we define the support of λ as $\text{supp}_\Omega \lambda = \{\tilde{\alpha} : c_{\tilde{\alpha}} \neq 0\}$. Also if $J \subset \tilde{\Delta} \setminus I$ we define $\bar{p}_J = \prod_{\tilde{\alpha} \in J} \bar{p}_{\tilde{\alpha}}$.

Proposition 5.1. *Let $J \subset \tilde{\Delta} \setminus I$ then the equation $\bar{p}_J = 0$ is reduced.*

Furthermore the divisor of the section \bar{p}_{I^c} is the complement of the unique open H^o orbit in G/P_I , and we have $H^o P_I = H P_I = \bar{H} P_I$.

Proof. We start with the first assertion. Let D_J denote the divisor of $\bar{p}_J = 0$ with reduced structure. Take $\lambda \in \Lambda_I^+$ with the property that $\mathcal{L}_\lambda \simeq \mathcal{O}(D_J)$ and let $\varphi \in \Gamma(G/P_I, \mathcal{L}_\lambda)$ such that $\text{div } \varphi = D_J$. We claim that $\lambda = \tilde{\omega}_J := \sum_{\tilde{\alpha} \in J} \tilde{\omega}_{\tilde{\alpha}}$.

Notice that by definition φ divides \bar{p}_J and for big enough n the section \bar{p}_J divides φ^n . So $\tilde{\omega}_J = \lambda + \mu$ and $n\lambda = \tilde{\omega}_J + \nu$ with μ and ν dominant. Moreover D_J is an \bar{H} invariant so φ is an eigenvector under the action of \bar{H} . In particular λ and also μ and ν are quasi spherical. Recall that $\tilde{\Delta}_{ne} = \{\tilde{\alpha} \in \Delta_1 : \alpha \text{ is not exceptional}\}$. We can write

$$\lambda = \sum_{\tilde{\alpha} \in \tilde{\Delta}_{ne} \setminus I} c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} + \sum_{\alpha \in \Delta_e : \tilde{\alpha} \notin I} c_\alpha \omega_\alpha.$$

Since φ divides \bar{p}_J we obtain $c_{\tilde{\alpha}}, c_\alpha \leq 1$ and $c_{\tilde{\alpha}} = c_\alpha = 0$ for $\tilde{\alpha} \notin J$. Also since \bar{p}_J divides φ^n we obtain $c_{\tilde{\alpha}}, c_\alpha \geq 1$ for $\tilde{\alpha} \in J$. So $c_\alpha = c_{\tilde{\alpha}} = 1$ if $\tilde{\alpha} \in J$ and $\lambda = \tilde{\omega}_J$ as claimed.

Now let $U = H^o \cdot [P_I]$ denote the open H^o orbit in G/P_I and U' the complement of the divisor D_{I^c} . Since U' is H^o stable $U \subset U'$.

We claim that U is affine. To see this is enough to prove that the Lie algebra of $H^o \cap P_I$ is reductive. Indeed we have that this Lie algebra is equal to the Lie algebra of L_I^σ where L_I is the Levi factor of P_I containing T .

Since U is affine, by [7] Proposition 3.1 pg. 66, $D = G/P_I \setminus U$ has pure codimension one. Let $\mathcal{L}_\lambda \simeq \mathcal{O}(D)$ and let $\varphi \in \Gamma(G/P_I, \mathcal{L}_\lambda)$ be such that $\operatorname{div} \varphi = D$.

Since $U \subset U'$ and $\bar{p}_{I^c} = 0$ is reduced we have that \bar{p}_{I^c} divides φ . So \mathcal{L}_λ is ample. Moreover the section φ must be an eigenvector under the action of \bar{H} so λ is quasi spherical. This together with the fact that $\lambda \in \Lambda_I$ easily implies that, up to a non zero constant,

$$\varphi = \prod_{\tilde{\alpha} \in I^c \cap \tilde{\Delta}_{ne}} \bar{p}_{\tilde{\alpha}}^{c_{\tilde{\alpha}}} \cdot \prod_{\alpha \in \Delta_e : \tilde{\alpha} \in I^c} \bar{q}_{\alpha}^{c_{\alpha}}$$

with the exponents $c_{\tilde{\alpha}}$ and c_{α} positive. On the other hand since φ is reduced we must have $c_{\tilde{\alpha}} = c_{\alpha} \leq 1$ for all $\tilde{\alpha}, \alpha$. So $\lambda = \tilde{\omega}_{I^c}$ and $\varphi = \bar{p}_{I^c}$ as claimed.

Finally since the complement of the section \bar{p}_J is stable by the action of H , or \bar{H} and is a single H^o orbit, it is also a single H or \bar{H} orbit. \square

5.2. Semistable points in a smooth toroidal compactification. In this section we prove that the set of semistable points in a smooth toroidal compactification does not depend on the choice of an ample line bundle.

We need to make few remarks on weights and convex functions. We start with a simple and well known Lemma on root systems.

Lemma 5.2. *Let $\{\alpha_1, \dots, \alpha_r\}$ be a set of simple roots in a root system R , and $\{\omega_1, \dots, \omega_r\}$ the corresponding set of fundamental weights. If $K \subset \{1, \dots, r\}$ every ω_j can be expressed as*

$$\omega_j = \sum_{h \in K} a_h \alpha_h + \sum_{k \notin K} b_k \omega_k$$

with a_h, b_k non negative rational numbers.

Furthermore if $K = \Delta$, and R is irreducible the a_h 's are strictly positive.

Proof. If $r = 1$ there is nothing to prove so we can proceed by induction.

Assume $|K| < r$. The space A and B respectively spanned by the α_h 's with $h \in K$ and by the ω_k 's with $k \notin K$ are mutually orthogonal. It follows that ω_j can be uniquely written as

$$\omega_j = \gamma_j + \delta_j$$

with $\gamma_j \in A$ and $\delta_j \in B$.

If $j \notin K$ then $\gamma_j = 0$ and there is nothing to prove.

If $j \in K$, γ_j is a fundamental weight for the root system in A having the α_h 's with $h \in K$ as simple roots. Thus by induction $a_h \geq 0$ for each $h \in K$. Write $\delta_j = \sum_{k \notin K} b_k \omega_k$. We get $b_k = \langle \delta_j, \alpha_k^\vee \rangle = -\langle \gamma_j, \alpha_k^\vee \rangle \geq 0$ as desired.

Assume now $|K| = r$. Write $\omega_j = \sum_{h=1}^r a_{j,h} \alpha_h$ and notice that $0 < (\omega_j, \omega_j) = a_{j,j}(\alpha_j, \omega_j)$. Since $(\alpha_j, \omega_j) > 0$ we deduce that $a_{j,j} > 0$. In particular we can write

$$\alpha_j = \frac{\omega_j}{a_{j,j}} + \xi_j$$

where ξ_j is a linear combination of the elements in $\Delta \setminus \{\alpha_j\}$. It follows that $\omega_k = (a_{k,j}/a_{j,j})\omega_j + x$ with x a linear combination of the elements in $\Delta \setminus \{\alpha_j\}$. Using the positivity of $a_{j,j}$ our claim now follows from the previous analysis applied in the case in which $K = \Delta \setminus \{\alpha_j\}$.

It remains to show that in the irreducible case $a_{j,h} \neq 0$ i.e. $(\omega_j, \omega_h) \neq 0$ for each $j, h = 1, \dots, r$. By contradiction assume that say $(\omega_1, \omega_2) = 0$. This means that ω_2 lies in the span of $\alpha_2, \dots, \alpha_r$ and it is a fundamental weight for the root system having these roots as a set of simple roots. Let R' be the irreducible component of this root system containing α_2 . We can assume that our ordering of simple roots is such that $\alpha_2, \dots, \alpha_s$ are a set of simple roots for R' . By induction $\omega_2 = \sum_{h=2}^s a_{2,h}\alpha_h$ with $a_{2,h} > 0$. On the other hand

$$0 = (\omega_2, \alpha_1) = \sum_{h=2}^s a_{2,h}(\alpha_h, \alpha_1)$$

so that $(\alpha_h, \alpha_1) = 0$ for each $h = 1, \dots, s$. This clearly contradicts the irreducibility of R . \square

Let Y be a smooth toroidal compactification of G/H and let F_Y be the associated decomposition of the Weyl cohamber C .

We define the support $\text{supp}_\Omega \rho$ of a face ρ of F_Y as

$$\text{supp}_\Omega \rho := \{\tilde{\alpha} \in \tilde{\Delta} \mid \tilde{\alpha} \text{ is not identically zero on } \rho\}.$$

Lemma 5.3. *Let $\underline{\lambda} = (\lambda_\tau)_{\tau \in F_Y(\ell)} \in \text{SPic}_0(Y)$ be such that $\mathcal{L}_{\underline{\lambda}}$ is ample. Let ρ be a face of F_Y . Then there exist a positive integer n and $\mu \in \Omega_H^+$ such that $\text{supp}_\Omega \mu = \text{supp}_\Omega \rho$ and $\mu \in \mathcal{A}(n\underline{\lambda})$.*

Proof. Let $S = \text{supp}_\Omega \rho$ and $T = \tilde{\Delta} \setminus \text{supp}_\Omega \rho$. Let τ be a maximal dimensional face containing ρ . Since $\mathcal{L}_{\underline{\lambda}}$ is ample, Corollary 3.9 iv) implies that λ_τ is regular dominant. So by Lemma 5.2 there exists a positive integer n such that we can write $n\lambda_\tau$ as

$$n\lambda_\tau = \sum_{\tilde{\alpha} \in S} a_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} + \sum_{\tilde{\alpha} \in T} b_{\tilde{\alpha}} \tilde{\alpha}$$

with $a_{\tilde{\alpha}}$ positive integers and $b_{\tilde{\alpha}}$ non negative integers. Set $\mu = \sum_{\tilde{\alpha} \in S} a_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}}$. We have $\mu = n\lambda_\tau$ on ρ and $\mu \leq n\lambda_\tau \leq n\underline{\lambda}$ on the Weyl cohamber C again by Corollary 3.9 iv). \square

The following Lemma is a sort of converse of Lemma 5.3.

Lemma 5.4. *Let $\underline{\lambda} = (\lambda_\tau)_{\tau \in F_Y(\ell)} \in \text{SPic}_0$ be such that $\mathcal{L}_{\underline{\lambda}}$ is ample. Let $\mu \in \Omega^+$ and n a positive integer with $\mu \in \mathcal{A}(n\underline{\lambda})$ and $\mu = n\underline{\lambda}$ on ρ . Then $\text{supp}_\Omega \mu \supset \text{supp}_\Omega \rho$.*

Proof. If ρ is the zero face there is nothing to prove. Assume that ρ has positive dimension. By eventually substituting $\underline{\lambda}$ with $n\underline{\lambda}$, let us also assume that $n = 1$.

Let $\rho(1)$ be the set of 1 dimensional faces contained in ρ and notice that $\text{supp}_\Omega \rho = \bigcup_{\vartheta \in \rho(1)} \text{supp}_\Omega \vartheta$. So it is enough to prove the claim in the case of one dimensional faces.

Let ρ be one dimensional and choose a non zero point v in ρ .

Take a face τ of maximal dimension containing ρ and define

$$\tau^\rho = \{u \in \Lambda_{\mathbb{R}}^\vee : v + t(v - u) \in \tau \text{ for some positive real number } t > 0\}.$$

Notice that $\mu \geq \lambda_\tau$ on τ^ρ . Indeed if $u \in \tau^\rho$ there is a positive t such that $v + t(v - u) \in \tau$. Since $\mu \in \mathcal{A}(\underline{\lambda})$ we have $\mu(v + t(v - u)) \leq \lambda_\tau(v + t(v - u))$. But $\underline{\lambda} = \mu$ on ρ so that $\mu(v) = \lambda_\tau(v)$, so $\mu(u) \geq \lambda_\tau(u)$.

Since the support of F_Y equals C it is then clear that

$$\bigcup_{\tau \in F_Y(\ell) : \tau \supset \rho} \tau^\rho = \{u \in \Lambda_{\mathbb{R}}^\vee : \langle \tilde{\alpha}, u \rangle \leq 0 \text{ for all } \tilde{\alpha} \notin \text{supp}_\Omega \rho\}.$$

Thus every $\tilde{\alpha} \in \text{supp}_\Omega \rho$ lies in at least one of the sets τ^ρ . It follows that

$$\langle \mu, \tilde{\alpha} \rangle \geq \langle \lambda_\tau, \tilde{\alpha} \rangle > 0$$

since $\underline{\lambda}$ is ample. Thus $\tilde{\alpha} \in \text{supp}_\Omega \mu$. \square

We can now characterize the set of semistable points with respect to an ample line bundle \mathcal{L} on Y .

Consider the G equivariant projection $\phi : Y \longrightarrow X$ onto the wonderful compactification X . For each $\mu \in \Omega$ we pull back the H^o invariant section p^μ . This is an H^o invariant section of $\Gamma(Y, \phi^*(\mathcal{L}_\mu))$ and we denote it by the same symbol p^μ . For any subset $I \subset \hat{\Delta}$ we set $p_I := \prod_{\tilde{\alpha} \in I} p^{\tilde{\alpha}}$. We also remark that the condition of being semistable does not depend on the group K between H^o and \bar{H} . Indeed, by the description of invariant sections, if f is a H^o invariant section which does not vanish on x there exists an integer n such that f^n is invariant under \bar{H} . In view of this we will speak of semistable points without specifying the group K .

Proposition 5.5. *Let Y be a smooth projective toroidal embedding of G/H and let \mathcal{L} be an ample line bundle on Y . Let ρ be a face of F_Y and let O_ρ be the corresponding G orbit. A point $x \in O_\rho$ is \mathcal{L} semistable if and only if $p_{\text{supp}_\Omega \rho}(x) \neq 0$.*

Proof. Let $\mathcal{L} = \mathcal{L}_{\underline{\lambda}}$. Let ρ be a face of F_Y and $x \in O_\rho$. Assume that $p_{\text{supp}_\Omega \rho}(x) \neq 0$. By Lemma 5.3 there exist a positive integer n and a dominant weight μ such that $\mu \in \mathcal{A}(n\underline{\lambda})$, $\mu = n\underline{\lambda}$ on ρ and $\text{supp}_\Omega \mu = \text{supp}_\Omega \rho$. Since p^μ is a product of the sections $p_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \text{supp}_\Omega \rho$ we have that $p^\mu(x) \neq 0$. Thus $s^{n\underline{\lambda} - \mu} p^\mu$ is an H invariant section of $\mathcal{L}_{\underline{\lambda}}^n$ not vanishing on x and x is semistable.

Conversely let x be semistable. Then by the description of invariants given in the proof of Theorem 4.13 there exists a positive integer n and a dominant spherical weight $\mu \in \mathcal{A}(n\underline{\lambda})$ such that $s^{n\underline{\lambda} - \mu} p^\mu(x) \neq 0$. The condition $s^{n\underline{\lambda} - \mu}(x) \neq 0$ implies $\mu = n\underline{\lambda}$ on ρ so we can apply Lemma 5.4

and we deduce that $\text{supp}_\Omega \mu \supset \text{supp}_\Omega \rho$. In particular $p_{\tilde{\alpha}}(x) \neq 0$ for all $\tilde{\alpha} \in \text{supp}_\Omega \rho$ or equivalently $p_{\text{supp}_\Omega \rho}(x) \neq 0$. \square

The above Proposition has the following

Corollary 5.6. *Let \mathcal{L} and \mathcal{L}' be two ample line bundles on Y . Then $Y^{ss}(\mathcal{L}) = Y^{ss}(\mathcal{L}')$.*

In view of this Corollary we shall from now on say that a point is semistable if it is semistable with respect to any ample line bundle and we shall denote the set of semistable points by Y^{ss} .

We now give a more set theoretic description of semistable points. Take a face ρ of the fan F_Y and denote by O_ρ the corresponding G orbit in Y . Set $I = \text{supp}_\Omega \rho$ and consider the G equivariant projection $\phi : Y \rightarrow X$. Remark that for any η in the relative interior of ρ the point y_η depends only on $\text{supp}_\Omega \rho$ and not on the choice of η . Thus we can denote this point by y_ρ . By definition for such one parameter subgroup η we have $\phi(y_\eta) = x_\eta$. In particular it follows that $\phi(O_\rho) = O_I$ the open orbit in X_I and that the projection $\phi : O_\rho \rightarrow O_I$ is a G equivariant fibration.

By [4] we have a G equivariant projection $\pi_I : O_I \rightarrow G/P_{I^c}$ with $I^c = \tilde{\Delta} \setminus I$. Composing we get a fibration

$$\gamma_I := \pi_I \circ \phi : O_\rho \rightarrow G/P_{I^c}$$

whose fiber over the point $\pi_I \circ \phi(y_\rho)$ we denote by F_ρ .

In view of Proposition 5.1 and Proposition 5.5 we immediately get

Proposition 5.7. *A point x in O_ρ is semistable if and only if its H orbit intersects F_ρ . So we have that $O_\rho^{ss} := Y^{ss} \cap O_\rho$ is equal to $H^o F_\rho$ (and also to $K F_\rho$ and $\bar{H} F_\rho$).*

Proof. This is clear since by Proposition 5.1 the section $p_{\text{supp}_\Omega \rho}$ does not vanish exactly on the inverse image under γ_I of the open H^o orbit in G/P_{I^c} . \square

By Proposition 5.7 we then have that $O_\rho^{ss} = H \times_{H \cap P_{I^c}} F_\rho$.

Let us now take a subset $J \subset \tilde{\Delta}$. We denote by L the standard Levi factor of P_J and recall that L is σ stable and that if U_{P_J} is the unipotent radical of P_J , $\sigma(U_{P_J}) = (U_{P_J})^-$ the opposite unipotent.

Lemma 5.8. *For any subset $J \subset \tilde{\Delta}$ we have*

$$K \cap L = K \cap P_J.$$

Proof. It is enough to analyze the case of $K = \bar{H}$. Since our problem is a problem of support we can work with the associated reduced subgroup $\bar{H}_{red} = \{x \in G : x\sigma(x)^{-1} \in Z(G)\}$.

Take $x = mu \in P_J \cap \bar{H}_{red}$ with $m \in L$ and $u \in U_{P_J}$. Then clearly $u\sigma(u)^{-1} \in L$. It follows that $\sigma(u)^{-1} \in U_{P_J}^- \cap P_J = \{1\}$ thus $u = 1$ and $x \in L$ as desired. \square

Going back to the semistable points in the orbit $O_\rho \subset Y$, we get

Proposition 5.9. *Let L be the standard Levi factor of P_{I^c} . Set $K_L = K \cap L$. We have a K equivariant isomorphism*

$$O_\rho^{ss} \simeq K \times_{K_L} F_\rho.$$

In particular this induces a closure preserving bijection between K orbits in O_ρ^{ss} and K_L orbits in F_ρ .

Now consider the fiber F_I of π_I containing $\phi(y_\rho)$. We know from [4] that the solvable radical of P_{I^c} acts trivially on F_I and that $F_I = \overline{L}/\overline{H}_{\overline{L}}$ where \overline{L} is the adjoint quotient of L and $\overline{H}_{\overline{L}}$ is the subgroup fixed by the involution induced by σ .

By the description of Y given in Theorem 3.5 it now follows that if we set L_ρ equal to the quotient of L modulo the subgroup in the center of L generated by the one parameter subgroups η with $\eta \in \rho$, F_ρ can be identified with L_ρ/H_ρ , where H_ρ is isogenous to the subgroup fixed by the involution on L_ρ induced by σ .

Under this identification S_{H_ρ} coincides with $Y_S \cap O_\rho$. We thus can apply Remark 4.11 and deduce

Proposition 5.10. *Let $x \in Y_S \cap O_\rho$. Then the orbit Kx is closed in O_ρ^{ss} .*

6. CLOSED SEMISTABLE ORBITS

In this section we prove that in the case of a smooth toroidal compactification Y of G/H the orbits through the elements of Y_S are the closed semistable orbits. In this way we give a geometric counterpart to Theorem 4.1.

We show first that the closure of K orbits does not interact with G orbits.

Proposition 6.1. *Let Y be a smooth toroidal compactification of G/H , let \mathcal{L} be an ample line bundle on Y . Let $x, y \in Y^{ss}(\mathcal{L})$. If $y \in \overline{K}x$ then $y \in G \cdot x$.*

Proof. As we have pointed out in Corollary 5.6, the set of semistable points of Y does not depend on the choice of the ample line bundle \mathcal{L} .

Let $\mathcal{L} = \mathcal{L}_\lambda$. For n big enough we have that $\mathcal{L}_{n\lambda + \alpha_D}$ is also ample for every $D \in \Delta_Y$. In particular for a large enough positive integer m , we can find invariant sections $f_D \in \Gamma(Y, \mathcal{L}_{m(n\lambda + \alpha_D)})^K$ and $f \in \Gamma(Y, \mathcal{L}_{mn\lambda})^K$ such that $f_D(y) \neq 0$ and $f(y) \neq 0$.

Set now

$$U = \{z \in Y \text{ such that } f(z) \neq 0 \text{ and } f_D(z) \neq 0 \text{ for all } D \in \Delta_Y\}.$$

The set U is an open affine H invariant subset of Y^{ss} with the property that if

$$\pi : Y^{ss} \rightarrow K \backslash Y$$

is the quotient morphism $U = \pi^{-1}(\pi(U))$. In particular $x \in U$. Furthermore on U the line bundles $\mathcal{L}_{m(n\lambda + \alpha_D)}$ and $\mathcal{L}_{mn\lambda}$ have a H equivariant trivialization.

It follows that also the line bundle $\mathcal{L}_{m\alpha_D} = \mathcal{L}_{m(n\lambda + \alpha_D)} \otimes \mathcal{L}_{mn\lambda}^{-1}$ has an H equivariant trivialization ϕ_D on U . Thus we can consider the \bar{H} invariant function $t_D = \phi_D(s_D^m)$ on U . Now by Theorem 3.5 a G orbit in Y is determined by the set of D such that s_D vanishes on the orbit. This implies our claim. \square

Remark 6.2. The following simple example shows that Proposition 6.1 does not hold for a non toroidal Y . Take the compactification $\mathbb{P}(\text{End}(\mathbb{C}^3))$ of $PSL(3)$, so $G = SL(3) \times SL(3)$ and K is the normalizer of the diagonal copy of $SL(3)$. The elements

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

give a counterexample to the statement in the Proposition.

Let U be any G stable open subset of smooth toroidal projective embedding Y . The proof of Proposition 6.1 implies

Proposition 6.3. $\pi^{-1}(\pi(U \cap Y^{ss})) = U \cap Y^{ss}$. Furthermore

$$\pi|_{U \cap Y^{ss}} : U \cap Y^{ss} \rightarrow \tilde{W} \backslash U_S$$

is a well defined quotient map.

Notice however that, as the following example shows, there are ample line bundles \mathcal{L} on Y such that if we restrict \mathcal{L} to U , $U^{ss}(\mathcal{L}) \neq U \cap Y^{ss}$. Indeed, take Y equal to the wonderful embedding of $PSL(3)$, $\mathcal{L} = \mathcal{L}_{\tilde{\omega}_1 + \tilde{\omega}_2}$ and U equal to the complement of the divisor of equation $s_{\tilde{\alpha}_1} = 0$. Then U is isomorphic to the open set in $\mathbb{P}(\text{End}(\mathbb{C}^3))$ of classes of matrices of rank at least 2 and there is an invariant namely $s_{\tilde{\alpha}_1}^{-1} p_{\tilde{\alpha}_1}^3$, which up to a constant gives the cube of the trace, defined on U and not vanishing on

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

while x is not semistable in Y .

From Propositions 5.10 and 6.1 we finally get

Theorem 6.4. *Let Y be a smooth toroidal compactification of G/H and let $H^o \subset K \subset \bar{H}$. Let $x \in Y^{ss}$ then Kx is closed in Y^{ss} if and only if $Kx \cap Y_S \neq \emptyset$.*

Proof. From the proof of Theorem 4.1 we get that $Y_S \subset Y^{ss}$ and the fact that if $x \in Y_S$ its orbit is closed is Proposition 5.10.

On the other hand since $K \backslash Y = \tilde{W} \backslash Y_S$, the restriction of the quotient map

$$\pi : Y^{ss} \rightarrow K \backslash Y$$

to Y_S is surjective. Given $p \in K \backslash Y$, $\pi^{-1}(p)$ contains a unique closed orbit and this by the first part is necessarily the orbit of a point in $\pi^{-1}(p) \cap Y_S$ \square

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